



Multi-Objective Decision Making

PUBLIC SECTOR TAX PLANNING

The government is a prime candidate for performing decision making that involves multiple (conflicting) objectives. Most governmental decisions have political ramifications, with each constituency benefiting along some dimensions but losing along others. A real example of such governmental decision making is discussed by Chrisman et al. (1989) in their paper “A Multiobjective Linear Programming Methodology for Public Sector Tax Planning.” During the 1980s, the city of Peoria, Illinois, experienced a deterioration of its industrial base. As a result, its population and property values declined, which meant a decrease in property taxes. Because Peoria had always relied heavily on property taxes, it became more difficult for it to generate sufficient revenue for existing public services.

In response, the city funded a management science study to determine new ways to generate revenue that would be politically acceptable. Several

forms of sales tax were suggested, such as a sales tax on everything but food, drugs, and durable goods, a sales tax on food and drugs, a sales tax on durable goods, and a sales tax on gasoline. Politically, the city had several objectives. It certainly needed to maintain its current level of tax revenue, but it also wanted to put less tax burden on low-income families, it wanted to reduce the property tax rate, and it wanted to keep businesses from moving out of Peoria. This suggested a multiple-objective linear programming (MOLP) approach. Specifically, the study identified four objectives: (1) reduce the property tax rate, (2) minimize the tax burden on low-income households, (3) minimize the flight of businesses and shoppers to the suburbs to escape higher sales taxes, and (4) minimize the gasoline tax rate.

Using these four objectives, the model then formulated a number of constraints, including (1) maintain the current level of tax revenues (at least), (2) ensure that the property tax rate does not increase, (3) ensure that the general sales tax rate excluding food, drugs, and durable goods is between 1% and 3%, and others. (Actually, as the city manager studied the model's output and saw results he didn't agree with, he suggested more constraints in order to produce different types of solutions. Remember that this took place in a *political* context!) The decision variables were the various tax rates, including the property tax rate and the proposed sales tax rates.

Unlike examples in previous chapters where there was a single objective, an MOLP model produces *many* solutions. The solutions worth examining are called **efficient solutions**, in the sense that any change in an efficient solution that makes one objective better automatically makes another objective worse. This means that the set of efficient solutions is the only set decision makers need to choose from, and they can make the choice by using their own preference function. This is indeed what happened in the Peoria study. After the city manager studied the set of efficient solutions, he was finally able to select a solution that he deemed best—it satisfied the constraints, it did well across the various objectives, and it was politically acceptable. Because of the political context, it might be hard to argue that the accepted solution was really “optimal,” but the MOLP approach did what it was supposed to do. It allowed the city to make a decision based on rational criteria, it illustrated the trade-offs between the objectives associated with planning the tax structure, and it decreased the property tax rate from 2.76% to 1.64% while maintaining the level of tax revenue. (See Problem 41 for an illustration of this problem.) ■

9.1 INTRODUCTION

In many of your classes, you have probably discussed how to make good decisions. Usually, you assume that the correct decision optimizes a *single* objective, such as profit maximization or cost minimization. In most situations you encounter in business and life, however, there is more than one relevant objective. For example, when you graduate you will (we hope) receive several job offers. Which should you accept? Before deciding which job offer to accept, you might consider how each job “scores” on several objectives, such as salary, interest in work, quality of life in the city you will live in, and nearness to family. In this situation it is difficult to combine your four objectives into a single objective. Similarly, in determining an optimal investment portfolio, you want to maximize expected return but you also want to minimize risk. How do you reconcile these conflicting objectives? In this chapter we will discuss three tools, **goal programming**, **trade-off curves**, and the **Analytic Hierarchy Process (AHP)**, that decision makers can use to solve multiple-objective problems. Fortunately, all three of these tools can be implemented fairly easily on a spreadsheet.

9.2 GOAL PROGRAMMING

In many situations a company wants to achieve several objectives. Given limited resources, it may prove impossible to meet all objectives simultaneously. If the company can prioritize its objectives, then **goal programming** can often be used to choose good decisions. The following media selection problem is typical of the situations in which goal programming is useful.

EXAMPLE

9.1

DETERMINING AN ADVERTISING SCHEDULE AT LEON BURNIT

The Leon Burnit Ad Agency is trying to determine a TV advertising schedule for a client. The client has three goals (listed here in descending order of importance). It wants its ads to be seen by

- Goal 1: at least 65 million high-income men (HIM)
- Goal 2: at least 72 million high-income women (HIW)
- Goal 3: at least 70 million low-income people (LIP)

Burnit can purchase several types of TV ads: ads shown on live sports shows, on game shows, on news shows, on sitcoms, on dramas, and on soap operas. At most \$700,000 total can be spent on ads. The advertising costs and potential audiences (in millions of viewers) of a 1-minute ad of each type are shown in Table 9.1. As a matter of policy, the client requires that at least two ads be placed on sports shows, on news shows, and on dramas. Also, it requires that no more than ten ads be placed on any single type of show. Burnit wants to find the advertising plan that best meets its client's goals.

TABLE 9.1 Data for Advertising Example

Ad Type	HIM	LIP	HIW	Cost
Sports show	7	4	8	\$120,000
Game show	3	5	6	\$40,000
News	6	5	3	\$50,000
Sitcom	4	5	7	\$40,000
Drama	6	8	6	\$60,000
Soap opera	3	4	5	\$40,000

Solution

First, we build a spreadsheet model to see whether all of the goals can be met simultaneously. In this spreadsheet model we must keep track of the following:

- the number of sports and soap opera ads placed
- the cost of the ads
- the number of exposures to each group (HIM, HIW, and LIP)
- the deviation from the exposure goal of each group



DEVELOPING THE SPREADSHEET MODEL

The spreadsheet model can be formed as follows. (See Figure 9.1 and the Model sheet of the file BURNIT1.XLS.)

- ❶ **Inputs.** Enter all inputs in the shaded ranges.
- ❷ **Numbers of ads.** Enter *any* trial values for the numbers of ads in the Ads range.
- ❸ **Total cost.** Calculate the total amount spent on ads in the TotCost cell with the formula

=SUMPRODUCT(UnitCosts,Ads)

- ❹ **Exposures obtained.** Calculate the number of people (in millions) in each group that the ads reach in the Obtained range. Specifically, enter the formula

=SUMPRODUCT(B7:G7,Ads)

in cell B26 for the HIM group, and copy this to the rest of the Obtained range for the other two groups.



USING THE SOLVER

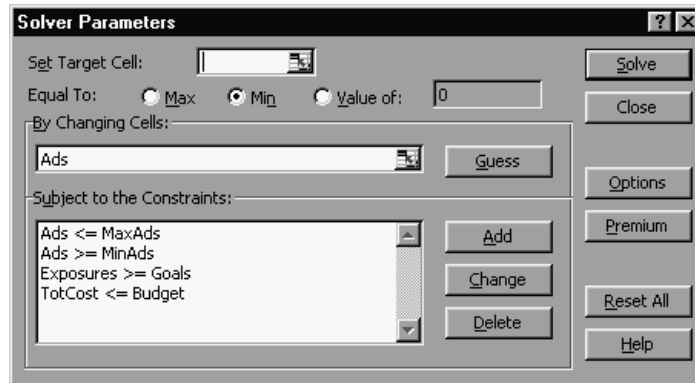
The completed Solver dialog box appears in Figure 9.2. At this point there is no objective to maximize or minimize. We are simply looking for *any* solution that meets all of the constraints. When we click on Solve, we get the message that there is no feasible solution. It is impossible to meet all of the client's goals and stay within this budget. To see how large the budget must be, we ran SolverTable with the Budget cell

	A	B	C	D	E	F	G
1	Advertising model - possible to meet all goals?						
2							
3	Note: All monetary values are in \$1000s, and all exposures to ads are in millions of exposures.						
4							
5	Exposures to various groups per unit of advertising						
6		Sports ad	Game show ad	News show ad	Sitcom ad	Drama ad	Soap opera ad
7	High-income men	7	3	6	4	6	3
8	High-income women	4	5	5	5	8	4
9	Low-income people	8	6	3	7	6	5
10							
11	Cost/unit	120	40	50	40	60	40
12							
13	Advertising plan						
14		Sports ad	Game show ad	News show ad	Sitcom ad	Drama ad	Soap opera ad
15	Minimum ads required	2	0	2	0	2	0
16		<=	<=	<=	<=	<=	<=
17	Number purchased	2,000	0,000	2,000	4,000	3,333	0,000
18		<=	<=	<=	<=	<=	<=
19	Maximum ads allowed	10	10	10	10	10	10
20							
21	Budget constraint	Total cost		Budget			
22		\$700	<=	\$700			
23							
24	Goals for numbers of exposures						
25		Exposures		Goal			
26	High-income men	62,000	>=	75			
27	High-income women	64,667	>=	72			
28	Low-income people	70,000	>=	70			
29							

Range names used:
 UnitCosts - B11:G11
 MinAds - B15:G15
 Ads - B17:G17
 MaxAds - B19:G19
 TotCost - B22
 Budget - D22
 Exposures - B26:B28
 Goals - D26:D28

FIGURE 9.1 Feasibility of Meeting All Goals

FIGURE 9.2
Solver Dialog
Box for Finding
a Feasible Solution



as the single input cell, varied from 700 to 850, and *any* cell as the output cell. The results appear in Figure 9.3. They show that unless the budget is greater than \$775,000, it is impossible to meet all of the client’s goals.

FIGURE 9.3
Checking How Large
the Budget Must Be

	A	B	C	D
30	Sensitivity to budget (to see how large it must be to meet all goals)			
31		\$B\$26		
32	700	Not feasible		
33	725	Not feasible		
34	750	Not feasible		
35	775	Not feasible		
36	800	75,000		
37	825	75,000		
38	850	75,000		

Now that we know that a \$700,000 budget is not sufficient to meet all of the client’s goals, we use goal programming to see how close Burnit can come to these goals. First, we introduce some terminology. The upper and lower limits on the ads of each type and the budget constraints are considered **hard** constraints in this model. This means that they cannot be violated under any circumstances. The goals on exposures, on the other hand, are considered **soft** constraints. The client certainly wants to satisfy these goals, but it is willing to come up somewhat short—in fact, it must because of the limited budget. In goal programming models the soft constraints are prioritized. We first try to satisfy the goals with the highest priority (in this case, HIM exposures). If there is still any room to maneuver, we then try to satisfy the goals with the next highest priority (HIW exposures). If there is *still* room to maneuver, we move on to the goals with the third highest priority, and so on.



DEVELOPING THE GOAL PROGRAMMING MODEL

In general, goal programming requires several consecutive Solver runs, one for each priority level. However, it is possible to set up the model so that we can make these consecutive runs with only minor modifications from one run to the next. We illustrate the procedure in Figure 9.4 on page 454. (See the Goals sheet of the file BURNIT1.XLS.) To develop this model, first make a copy of the original Model sheet shown in Figure 9.1. Then modify it using the following steps.

1 New changing cells. The exposure constraints are no longer shown as “hard” constraints. Instead, we introduce changing cells in the DevUnder and DevOver ranges (Dev is short for “deviations”) to indicate how much under or over each goal we are.

	A	B	C	D	E	F	G
24	Goals for numbers of exposures						
25		Exposures	Amt under goal	Amt over goal	Balance		Goal
26	High-income men	65,000	0	0	65,000	=	65
27	High-income women	60,250	11.75	0	72,000	=	72
28	Low-income people	58,750	11.25	0	70,000	=	70
29							
30	Deviations from goals (amounts below goals, or 0 if currently meeting goal)						
31		Deviation		Already obtained			
32	High-income men	0.000	<=	65,000			
33	High-income women	11,750	<=	72,000			
34	Low-income people	11,250	<=	70,000			
35							
36	<div style="border: 1px solid black; padding: 5px;"> Extra range names used: DevUnder - C26:C28 DevOver - D26:D28 Balances - E26:E28 Goals - G26:G28 DevUnder1 - B32:B34 Obtained - D32:D34 Dev1 - B32 Dev2 - B33 Dev3 - B34 </div>						
37							
38							
39							
40							
41							
42							
43							
44							
45							
46							

FIGURE 9.4 Minimizing Deviation for Highest Priority Goal

Enter any values in these ranges. (We entered 0's.) Note that in the Solver solution, at least one of these two types of deviations will always be 0 for each goal—we will either be below the goal or above the goal, but not both.

2 Balance equations. To tie these new changing cells to the rest of the model, we create “balances” in column E that must logically equal the goals in column G. To do this, enter the formula

$$=B26+C26-D26$$

in cell E26 and copy it down. The logical balance equation for each group specifies that the actual number of exposures, plus the number under the goal, minus the number over the goal, *must* equal the goal.

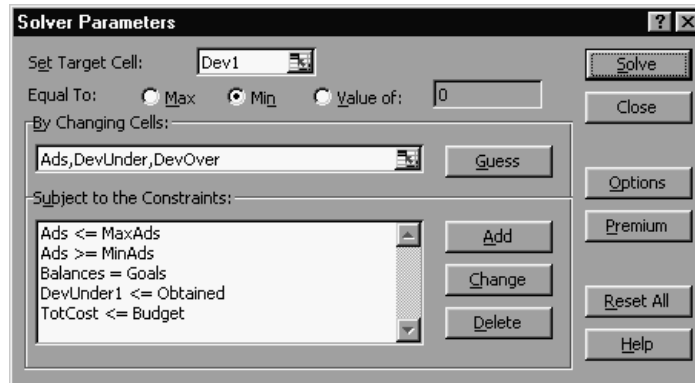
3 Constraints on deviations under. The client is concerned only with too *few* exposures, not with too many. Therefore, we set up constraints on the “under” deviations in rows 32–34. On the left side, in column B, enter links to the DevUnder range by entering the formula

$$=C26$$

in cell B32 and copying down. (We have named this range DevUnder1 because it is basically a copy of the DevUnder range.)

4 Highest priority goal. The first Solver run will try to achieve the highest priority goal (HIM exposures). To do so, we minimize the Dev1 cell. Do this as shown in Figure 9.4. Then set up the Solver dialog box as shown in Figure 9.5. (Also, check the Assume Linear Model and Assume Non-Negative boxes.) The constraints include the hard constraints, the balance constraint, and the DevUnder1 <= Obtained constraint. Note that we have entered the goals themselves in the Obtained range. Therefore, the DevUnder1 <= Obtained constraint at this point is essentially redundant—the “under” deviations cannot possibly be greater than the goals themselves. (Do you see why?) We include it because it will become important in later Solver runs, which will then require only minimal modifications. The solution from this Solver run is the one shown

FIGURE 9.5
Solver Dialog Box
for Highest Priority
Goal



in Figure 9.4. It shows that Burnit can satisfy the HIM goal completely. However, the other two goals are not satisfied because their “under” deviations are positive.

5 Second highest priority goal. Now we come to the key aspect of goal programming. Once a high priority goal is satisfied as fully as possible, we move on to the next highest priority goal. However, we do not want to lose what we already gained with the high priority goal. Therefore, we constrain its “under” deviation to be no greater than what we have already achieved. In this case we achieved a deviation of 0 in step 4, so enter 0 in cell D32 for the upper limit of the HIM “under” deviation. Then run the Solver again, changing only one thing in the Solver dialog box—make the Dev2 cell the target cell. Effectively, we are constraining the “under” deviation for the HIM group to remain at 0, and we are then minimizing the “under” deviation for the HIW group. The solution from this second Solver run appears in Figure 9.6. As we promised, the HIM goal has not suffered at all, but we are now a little closer to the HIW goal than before. It was under by 11.75 before, and now it is under by only 11. The lowest priority goal (for the LIP group) essentially “comes along for the ride” in this step. It could either improve or get worse. It happened to get worse, moving from under by 11.25 to under by 18.

	A	B	C	D	E	F	G
24	Goals for numbers of exposures						
25		Exposures	Amt under goal	Amt over goal	Balance	=	Goal
26	High-income men	65.000	0	0	65.000	=	65
27	High-income women	61.000	11	0	72.000	=	72
28	Low-income people	52.000	18	0	70.000	=	70
29							
30	Deviations from goals (amounts below goals, or 0 if currently meeting goal)						
31		Deviation		Already obtained			
32	High-income men	0.000	<=	0.000	Initially, enter large values in these cells (such as the original goals). Then, as high priority goals are met or partially met, enter the actual deviations obtained here (one at a time).		
33	High-income women	11.000	<=	11.000			
34	Low-income people	18.000	<=	70.000			
35							
36	Extra range names used:						
37	DevUnder - C26:C28						
38	DevOver - D26:D28						
39	Balances - E26:E28						
40	Goals - G26:G28						
41	DevUnder1 - B32:B34						
42	Obtained - D32:D34						
43	Dev1 - B32						
44	Dev2 - B33						
45	Dev3 - B34						
46							

FIGURE 9.6 Minimizing Deviation for Second Priority Goal

6 Lowest priority goal. You can probably guess the last step by now. We minimize the Dev3 cell, the deviation for the LIP group, while ensuring that the two higher priority goals are achieved as fully as in steps 4 and 5. As the model is set up, only two changes are necessary—enter 11 in cell D33 and change the Solver target cell to the Dev3 cell. When you run Solver this time, however, you will find that there is no room left to maneuver. The solution remains exactly the same as in Figure 9.6. This occurs frequently in goal programming models. After satisfying the first goal or two as fully as possible, there is often no room to improve later goals.

To summarize Burnit’s situation, the budget of \$700,000 allows it to satisfy the client’s HIM goal, miss the HIW goal by 11 million, and miss the LIP goal by 18 million. Given the priorities on these three goals, this is the best possible solution.

Sensitivity Analysis Sensitivity analysis should be a part of goal programming just as it is for previous models we have discussed. However, there is no quick way to do it (at least none we have thought of). SolverTable works on only a *single* objective, whereas goal programming requires a sequence of objectives. Therefore, if we wanted to see how the solution to Burnit’s model changes with different budgets, say, we would need to go through the above steps several times and keep track of the results manually. This is certainly possible, but it is tedious.

Effect of changing priorities With three goals, there are six possible orderings of the goals. The goal programming solutions corresponding to these orderings are listed in Figure 9.7. Row 4 corresponds to the ordering we used in the example. Clearly, the solution can change if the priorities of the goals change. For example, when we give the HIW goal the highest priority (rows 6, 7), *none* of the goals are achieved completely. (Problem 1 asks you to verify the details.)

	A	B	C	D	E	F	G	H	I	J	K	L
1	Results from changing priorities											
2												
3	Priority 1	Priority 2	Priority 3	Dev1	Dev2	Dev3	Sports ad	Game show ad	News show ad	Sitcom ad	Drama ad	Soap opera ad
4	HIM	HIW	LIP	0	11	18	2	0	5	0	3.5	0
5	HIM	LIP	HIW	0	11.75	11.25	2	0	5	2.25	2	0
6	HIW	HIM	LIP	3	6	12	2	0	2	0	6	0
7	HIW	LIP	HIM	3	6	12	2	0	2	0	6	0
8	LIP	HIM	HIW	1.956	9.304	0	2	0	3.043	4.696	2	0
9	LIP	HIW	HIM	3	7.333	0	2	0	2	4	3.333	0

FIGURE 9.7 Effect of Changing Priorities

MODELING ISSUES

1. The results for the Burnit model are based on allowing the numbers of ads to have noninteger values. However, they could easily be constrained to integer values, and the solution method would remain exactly the same.
2. Each priority level in the Burnit model contains exactly one goal. It is easy to generalize to the case where a given priority level can have multiple goals, each modeled with a certain deviation from a target. When we run the Solver for this priority level, we would use a weighted average of these deviations as the objective to minimize.
3. All of the deviations in the objectives of the Burnit model are “under” deviations. However, it is certainly possible to include “over” deviations as objectives. For

example, if the budget constraint were treated as a soft constraint, we would try to minimize its “over” deviation to stay as little over the budget as possible. It is even possible for *both* the “under” and “over” deviations of some goal to be included as objectives. This would occur in a situation where we want to come as close as possible to some target value—neither under nor over.

4. The use of changing cells for the “under” and “over” deviations might not be intuitive. However, it serves two purposes. First, it provides exactly the information we need for the objectives in goal programming. Second, it keeps the model linear. We tried using an IF function instead (without the under and over cells) to capture the “under” deviations. It looked great, but even when we unchecked the Assume Linear Model box, the Solver obtained the wrong answer! ■

ADDITIONAL APPLICATIONS

Goal programming has been applied to many situations where multiple objectives are present. Here are some examples.

- Optimally blending sausage. The conflicting objectives are cost, protein content, and fat content. See Problem 6 and Steuer (1984).
- Admitting students to a college. The conflicting objectives involve class size, diversity, in-state versus out-of-state mix, and academic quality of the class. See Problem 10 and Lee and Moore (1974).
- Assigning different types of employees (partners, junior associates, and so on) to different jobs at a CPA firm. The conflicting objectives are monthly billings and the number of workers of each type that the company can hire. See Problem 7 and Welling (1977).
- Determining the projects that a manufacturing firm should implement. The conflicting objectives include return on investment, cost, technological risk, and productivity improvement.
- Determining the type of recreation facilities that a city should build. The conflicting objectives include limitations on land use and maintenance cost, as well as desired usage of each facility. See Problem 13 and Taylor and Keown (1984).
- Determining the optimal product mix. The conflicting objectives include profit, maintaining a desired cash balance, and maintaining a desired value of the firm’s current ratio. See Problem 42 and Sartoris and Spruill (1974).
- Determining the optimal set of ingot molds to use for steel production. Bethlehem Steel did this. Their primary goal was to minimize the number of molds used for steel production. Their secondary goal was to minimize the weight of the steel that needed to be produced to fill the company’s orders. See Vasko et al. (1955) and Problem 18. ■

PROBLEMS

Skill-Building Problems

1. For each set of priorities of goals, solve the Leon Burnit problem and verify that the values in Figure 9.7 are correct.
2. Gotham City must determine how to allocate ambulances during the next year. It costs \$5000 per year to run an ambulance. Each ambulance must be assigned to one of two districts. Let x_i be the number of ambulances assigned to district i , $i = 1, 2$. The

average time (in minutes) it takes for an ambulance to respond to a call from district 1 is $40 - 3x_1$; for district 2 the time is $50 - 4x_2$. Gotham City has three goals (listed in order of priority):

- Goal 1: At most \$100,000 per year should be spent on ambulance service.
 - Goal 2: Average response time in district 1 should be at most five minutes.
 - Goal 3: Average response time in district 2 should be at most five minutes.
- a. Use preemptive goal programming to determine how many ambulances to assign to each district.
 - b. How would your answer change if goal 2 had the highest priority, then goal 3, and then goal 1?
3. Fruit Computer Company is ready to make its annual purchase of computer chips. Fruit can purchase chips (in lots of 100) from three suppliers. Each chip's quality is rated as excellent, good, or mediocre. During the coming year Fruit needs 5000 excellent chips, 3000 good chips, and 1000 mediocre chips. The characteristics of the chips purchased from each supplier are shown in Table 9.2. Each year, Fruit has budgeted \$28,000 to spend on chips. If Fruit does not obtain enough chips of a given quality, it can special-order additional chips at \$10 per excellent chip, \$6 per good chip, and \$4 per mediocre chip. Fruit assesses a penalty of \$1 for each dollar it goes over the annual budget (in payments to suppliers). Determine how Fruit can minimize the penalty associated with meeting the annual chip requirements. Also use preemptive goal programming to determine a purchasing strategy. Let the budget constraint have the highest priority, followed in order by the restrictions on excellent, good, and mediocre chips.
4. Hiland Appliance must determine how many color TVs and VCRs to stock. It costs Hiland \$300 to purchase a color TV and \$200 to purchase a VCR. A color TV requires three square yards of storage space, and a VCR requires one square yard. The sale of a color TV earns Hiland a profit of \$150, and each VCR sale earns a profit of \$100. Hiland has set the following goals (listed in order of importance):
- Goal 1: A maximum of \$20,000 can be spent on purchasing color TVs and VCRs.
 - Goal 2: Highland should earn at least \$11,000 profit from the sale of color TVs and VCRs.

- Goal 3: Color TVs and VCRs should not use up more than 200 square yards of storage space.

Use a preemptive goal programming model to determine how many color TVs and VCRs Hiland should order. How would the preemptive goal formulation be modified if Hiland's second goal were to have a profit of *exactly* \$11,000?

5. Stockco produces two products. Relevant information for each product is shown in Table 9.3. Stockco has a goal of \$48 in profits and incurs a \$1 penalty for each dollar it falls short of this goal. A total of 32 hours of labor are available. A \$2 penalty is incurred for each hour of overtime (labor over 32 hours) used, and a \$1 penalty is incurred for each hour of available labor that is unused. Marketing considerations require that at least seven units of product 1 be produced and at least ten units of product 2 be produced. For each unit (of either product) by which production falls short of demand, a penalty of \$5 is assessed.

TABLE 9.3 Data for Stockco Problem

	Product 1	Product 2
Labor hours required	4	2
Contribution to profit	\$4	\$2

- a. Determine how to minimize the total penalty incurred by Stockco.
 - b. Suppose the company sets (in order of importance) the following goals:
 - Goal 1: Avoid underutilization of labor.
 - Goal 2: Meet demand for product 1.
 - Goal 3: Meet demand for product 2.
 - Goal 4: Do not use any overtime.
 Use preemptive goal programming to determine an optimal production schedule.
6. Based on Steuer (1984). Deancorp produces sausage by blending beef head, pork chuck, mutton, and water. The cost per pound, fat per pound, and protein per pound for these ingredients are listed in Table 9.4. Deancorp needs to produce 100 pounds of sausage and has set the following goals, listed in order of priority:
- Goal 1: Sausage should consist of at least 15% protein.

TABLE 9.2 Data for Fruit Computer Problem

	Number of Chips per Lot of 100			Price per 100
	Excellent	Good	Mediocre	
Supplier 1	60	20	20	\$400
Supplier 2	50	35	15	\$300
Supplier 3	40	20	40	\$250

- Goal 2: Sausage should consist of at most 8% fat.
- Goal 3: Cost per pound of sausage should not exceed \$0.08.

Use preemptive goal programming model to determine the composition of sausage.

TABLE 9.4 Data for Deancorp Sausage Problem

	Beef Head	Pork Chuck	Mutton	Water
Fat (per lb)	0.05	0.24	0.11	0
Protein (per lb)	0.20	0.26	0.08	0
Cost (in cents)	12	9	8	0

7. Based on Welling (1977). The Touche Young accounting firm must complete three jobs during the next month. Job 1 will require 500 hours of work, job 2 will require 300 hours, and job 3 will require 100 hours. At present the firm consists of five partners, five senior employees, and five junior employees, each of whom can work up to 40 hours per month. The dollar amount (per hour) that the company can bill depends on the type of accountant assigned to each job, as shown in Table 9.5. (The “X” indicates that a junior employee does not have enough experience to work on job 1.) All jobs must be completed. Touche Young has also set the following goals, listed in order of priority:
- Goal 1: Monthly billings should exceed \$74,000.
 - Goal 2: At most one partner should be hired.
 - Goal 3: At most three senior employees should be hired.
 - Goal 4: At most one junior employee should be hired.

Use preemptive goal programming to help Touche solve its problem.

TABLE 9.5 Data for Touche Young Accounting Problem

	Job 1	Job 2	Job 3
Partner	160	120	110
Senior employee	120	90	70
Junior employee	X	50	40

8. There are four teachers in the Faber College Business School. Each semester, 200 students take each of the following courses: Marketing, Finance, Production, and Statistics. The “effectiveness” of each teacher in teaching each course is given in Table 9.6. Each teacher can teach a total of 200 students during the semester. The dean has set a goal of obtaining an average teaching effectiveness level of at least 6 in

each course. Deviations from this goal in any course are considered equally important. Determine the semester’s teaching assignments.

TABLE 9.6 Data for Teaching Problem

	Marketing	Finance	Production	Statistics
Teacher 1	7	5	8	2
Teacher 2	7	8	9	4
Teacher 3	3	5	7	9
Teacher 4	5	5	6	7

9. There are 17 neighborhoods in the city of Bloomington. The number of high school students in each neighborhood and the time required to drive from each neighborhood to each of the city’s two high schools (North and South) are listed in Table 9.7. The Bloomington Board of Education needs to determine how to assign students to high schools. All students in a given neighborhood must be assigned to the same high school. The Board has set (in order of priority, from highest to lowest) the following goals:
- Ensure that the difference in enrollment at the two high schools differs by at most 50.
 - Ensure that average student travel time is at most 13 minutes.
 - Ensure that at most 4% of the students must travel at least 25 minutes to school.

TABLE 9.7 Data for Bloomington High School Problem

Neighborhood	Number of Students	Minutes to South	Minutes to North
1	200	8	30
2	200	8	35
3	180	4	27
4	100	6	25
5	150	4	22
6	220	10	26
7	150	11	32
8	180	13	34
9	150	12	28
10	210	15	32
11	220	16	19
12	230	20	18
13	120	12	24
14	210	19	16
15	180	18	15
16	190	32	4
17	160	28	3

- a. Determine an optimal assignment of students to high schools.
- b. If the difference between enrollment at the two high schools could differ by at most 100 (a change in goal 1), how would your answer change?

Skill-Extending Problems

10. Based on Lee and Moore (1974). Faber College is admitting students for the class of 2001. Data on its applicants are shown in Table 9.8. It has set four goals for this class, listed in order of priority:
- Goal 1: The entering class should include at least 5000 students.
 - Goal 2: The entering class should have an average SAT score of at least 640.
 - Goal 3: The entering class should consist of at least 25% out-of-state students.
 - Goal 4: At least 2000 members of the entering class should not be business majors.

Use preemptive goal programming to determine how many applicants of each type to admit. Assume that all applicants who are admitted will decide to attend Faber.

TABLE 9.8 Data on Faber's Applicants

	SAT Score	Business Majors	Non-Business Majors
In-state	700	1500	400
In-state	600	1300	700
In-state	500	500	500
Out-of-state	700	350	50
Out-of-state	600	400	400
Out-of-state	500	400	600

11. During the next four quarters, Wivco faces the following demands for globots: quarter 1, 13; quarter 2, 14; quarter 3, 12; quarter 4, 15. Globots can be produced by regular-time labor or by overtime labor. Production capacity (number of globots) and production costs during the next four quarters are shown in Table 9.9. Wivco has set the following goals in order of importance:

TABLE 9.9 Data for Wivco Production Problem

	Regular Time		Overtime	
	Capacity	Cost per Unit	Capacity	Cost per Unit
Quarter 1	9	\$4	5	\$6
Quarter 2	10	\$4	5	\$7
Quarter 3	11	\$5	5	\$8
Quarter 4	12	\$6	5	\$9

- Goal 1: Each quarter's demand should be met on time.
- Goal 2: Inventory at the end of each quarter should not exceed three units.
- Goal 3: Total production cost should be no greater than \$250.

Use a preemptive goal programming model to determine Wivco's production schedule for the next four quarters. Assume that at the beginning of the first quarter, one globot is in inventory.

12. Lucy's Music Store at present employs five full-time employees and three part-time employees. The normal workload is 40 hours per week for full-time employees and 20 hours per week for part-time employees. Each full-time employee is paid \$6 per hour for work up to 40 hours per week and can sell five recordings per hour. A full-time employee who works overtime is paid \$10 per hour. Each part-time employee is paid \$3 per hour and can sell three recordings per hour. It costs Lucy \$6 to buy a recording, and each recording sells for \$9. Lucy has weekly fixed expenses of \$500. She has established the following weekly goals, in order of priority:
- Goal 1: Sell at least 1600 recordings per week.
 - Goal 2: Earn a profit of at least \$2200 per week.
 - Goal 3: Full-time employees should work at most 100 hours of overtime.
 - Goal 4: To promote a sense of job security, the number of hours by which each full-time employee fails to work 40 hours should be minimized.

Use a preemptive goal programming model to determine how many hours per week each employee should work.

13. Based on Taylor and Keown (1984). Gotham City is trying to determine the type and location of recreational facilities to build during the next decade. Four types of facilities are under consideration: golf courses, swimming pools, gymnasiums, and tennis courts. Six sites are under consideration. If a golf course is built, it must be built at either site 1 or site 6. Other facilities can be built at sites 2–5. The amounts of available land (in thousands of square feet) at sites 2–5 are given in Table 9.10. The cost of building each facility (in thousands of dollars), the annual maintenance cost

TABLE 9.10 Available Land for Gotham Problem

	Site			
	2	3	4	5
Available land	70	80	95	120

(in thousands of dollars) for each facility, and the land (in thousands of square feet) required for each facility are given in Table 9.11. The number of user days (in thousands) for each type of facility, shown in Table 9.12, depends on where it is built.

TABLE 9.11 Costs and Land Requirements for Gotham Problem

	Construction Cost	Maintenance Cost	Land Required
Golf	\$340	\$80	Not relevant
Swimming	\$300	\$36	29
Gymnasium	\$840	\$50	38
Tennis courts	\$85	\$17	45

TABLE 9.12 User Days for Gotham Problem

	Site					
	1	2	3	4	5	6
Golf	31	X	X	X	X	27
Swimming	X	25	21	32	32	X
Gymnasium	X	37	29	28	38	X
Tennis courts	X	20	23	22	20	X

- a. Consider the following set of priorities:
- Priority 1: The amount of land used at each site should be no greater than the amount of land available.
 - Priority 2: Construction costs should not exceed \$1.2 million.
 - Priority 3: User days should exceed 200,000.

- Priority 4: Annual maintenance costs should not exceed \$200,000.

For this set of priorities, use preemptive goal programming to determine the type and location of recreation facilities in Gotham City.

- b. Consider the following set of priorities:
- Priority 1: The amount of land used at each site should be no greater than the amount of land available.
 - Priority 2: User days should exceed 200,000.
 - Priority 3: Construction costs should not exceed \$1.2 million.
 - Priority 4: Annual maintenance costs should not exceed \$200,000.

For this set of priorities, use preemptive goal programming to determine the type and location of recreation facilities in Gotham City.

14. A small aerospace company is considering eight projects:
- Project 1: Develop an automated test facility.
 - Project 2: Bar code all inventory and machinery.
 - Project 3: Introduce a CAD/CAM system.
 - Project 4: Buy a new lathe and deburring system.
 - Project 5: Institute an FMS (Flexible Manufacturing System).
 - Project 6: Install a LAN (Local Area Network).
 - Project 7: Develop an AIS (Artificial Intelligence Simulation).
 - Project 8: Set up a TQM (Total Quality Management) program.

Each project has been rated on five attributes: return on investment (ROI), cost, productivity improvement, workforce requirements, and degree of technological risk. These ratings are given in Table 9.13. The company has set the following five goals (listed in order of priority):

- Goal 1: Achieve an ROI of at least \$3250.
- Goal 2: Limit cost to \$1300.
- Goal 3: Achieve a productivity improvement of at least 6.
- Goal 4: Limit workforce use to 108.
- Goal 5: Limit technological risk to a total of 4.

Use preemptive goal programming to determine which projects should be undertaken.

TABLE 9.13 Ratings for Aerospace Problem

	Project							
	1	2	3	4	5	6	7	8
Return on investment (ROI)	\$2070	\$456	\$670	\$350	\$495	\$380	\$1500	\$480
Cost	\$900	\$240	\$335	\$700	\$410	\$190	\$500	\$160
Productivity improvement	3	2	2	0	1	0	3	2
Workforce requirements	18	18	27	36	42	6	48	24
Degree of technological risk	3	2	4	1	1	0	2	3

15. A new president has just been elected and has set the following economic goals (listed from highest to lowest priority):

- Goal 1: Balance the budget (this means revenues are at least as large as costs).
- Goal 2: Cut spending by at most \$150 billion.
- Goal 3: Raise at most \$550 billion in taxes from the upper class.
- Goal 4: Raise at most \$350 billion in taxes from the lower class.

Currently the government spends \$1 trillion per year. Revenue can be raised in two ways: through a gas tax and through an income tax. You must determine: G , the per-gallon tax rate (in cents); T_1 , the tax rate charged on the first \$30,000 of income; T_2 , the tax rate charged on any income earned over \$30,000; and C , the cut in spending (in billions). If the government chooses G , T_1 , and T_2 , then we assume that the revenue given in Table 9.14 (in billions of dollars) is raised. Of course, the tax rate on income over \$30,000 must be at least as large as the tax rate on the first \$30,000 of income. Use preemptive goal programming to help the president meet his goals.

TABLE 9.14 Data for Government Economy Problem

	Low Income	High Income
Gas tax	G	$0.5G$
Tax on income up to \$30,000	$20T_1$	$5T_1$
Tax on income over \$30,000	0	$15T_2$

16. The HAL computer must determine which of eight research and development (R&D) projects to undertake. For each project, four quantities are of interest: (1) the net present value (NPV, in millions

TABLE 9.15 Data for R&D Problem

	NPV (millions)	Annual Growth Rate	Probability of Success	Cost (millions)
Project 1	\$40	20%	0.75	\$220
Project 2	\$30	16%	0.70	\$140
Project 3	\$60	12%	0.75	\$280
Project 4	\$45	8%	0.90	\$240
Project 5	\$55	18%	0.65	\$300
Project 6	\$40	18%	0.60	\$200
Project 7	\$40	18%	0.60	\$200
Project 8	\$90	19%	0.65	\$440

of dollars) of the project; (2) the annual growth rate in sales generated by the project; (3) the probability that the project will succeed; and (4) the cost (in millions of dollars) of the project. The relevant information is given in Table 9.15. HAL has set the following four goals:

- Goal 1: The total NPV of all chosen projects should be at least \$200 million.
- Goal 2: The average probability of success for all projects chosen should be at least 0.75.
- Goal 3: The average growth rate of all projects chosen should be at least 15%.
- Goal 4: The total cost of all chosen projects should be at most \$1 billion.

For the following sets of priorities, use preemptive (integer) goal programming to determine the projects that should be selected.

- a. Goal 2, Goal 4, Goal 1, Goal 3.
- b. Goal 1, Goal 3, Goal 4, Goal 2.

17. Based on Klingman and Phillips (1984). The Marines need to fill three types of jobs in two cities (Los Angeles and Chicago). The numbers of jobs of each type that must be filled in each city are shown in Table 9.16. The Marines available to fill these jobs have been classified into six groups according to the types of jobs each person is capable of doing, the type of job each person prefers, and the city in

TABLE 9.16 Marine Jobs to Be Filled

	Job Type	Jobs to Fill
Los Angeles	1	1000
Los Angeles	2	2000
Los Angeles	3	1500
Chicago	1	2000
Chicago	2	1000
Chicago	3	1000

which each person prefers to live. The data for each of these six groups are shown in Table 9.17. The Marines have the following three goals, listed from highest priority to lowest priority:

- Ensure that all jobs are filled by qualified workers.
- Ensure that at least 8000 employees are assigned to the jobs they prefer.
- Ensure that at least 8000 employees are assigned to their preferred cities.

Determine how the Marines should assign their workers. (*Note:* You may allow fractional assignments of workers.)

18. Based on Vasko et al. (1955). Bethlehem Steel can fill orders using five different types of steel molds. Up to three different molds of each type can be purchased. Each individual mold can be used to fill up to 100 orders per year. Six different types of orders must be filled during the coming year. The waste (in tons) incurred if a type of mold is used to fill an order is shown in Table 9.18 (where an “x” indicates that a type of mold cannot be used to fill an order). The number of each order type that must be filled during the coming year is shown in Table 9.19. Bethlehem Steel has the following two goals, listed in order of priority.
- Since molds are very expensive, Bethlehem wants to use at most five molds.
 - Bethlehem wants to have at most 600 tons of total waste.

Use preemptive goal programming to determine how Bethlehem should fill the coming year’s orders.

TABLE 9.18 Waste in Bethlehem Problem

	Order Type					
	1	2	3	4	5	6
Mold 1	1	2	3	4	5	6
Mold 2	x	1	2	3	4	5
Mold 3	x	x	1	2	3	4
Mold 4	x	x	x	1	2	3
Mold 5	x	x	x	x	1	2

TABLE 9.19 Number of Orders for Bethlehem Problem

	Number of Orders
Order 1	80
Order 2	60
Order 3	40
Order 4	100
Order 5	120
Order 6	140

TABLE 9.17 Group Data for Marine Problem

	Number of People	Jobs They Can Do	Preferred Job	Preferred City
Group 1	1500	1, 2	1	Los Angeles
Group 2	1500	2, 3	2	Los Angeles
Group 3	1500	1, 3	1	Chicago
Group 4	1500	1, 3	3	Chicago
Group 5	1500	2, 3	3	Los Angeles
Group 6	1500	3	3	Chicago

9.3 PARETO OPTIMALITY AND TRADE-OFF CURVES

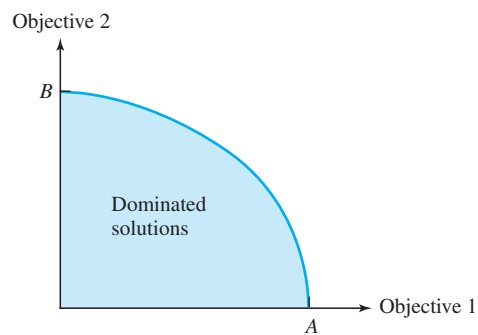
In a multiple-objective problem with no uncertainty, we often search for Pareto optimal solutions. We will assume that the decision maker has exactly two objectives and that the set of feasible points under consideration must satisfy a prescribed set of constraints.

First, we need to define some terms. A solution (call it A) to a multiple-objective problem is called **Pareto optimal** if no other feasible solution is at least as good as A with respect to every objective and strictly better than A with respect to at least one objective. A related concept is **domination**. We say a feasible solution B **dominates**

a feasible solution A to a multiple-objective problem if B is at least as good as A with respect to every objective and is strictly better than A with respect to at least one objective. From this definition, it follows that Pareto optimal solutions are feasible solutions that are not dominated.

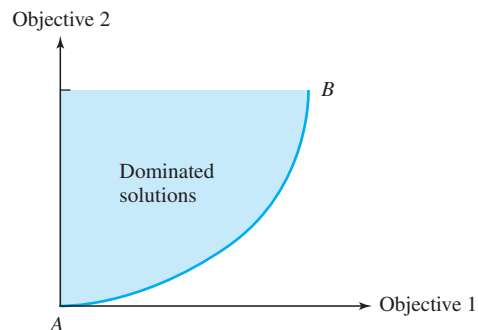
If we graph the “score” of all Pareto optimal solutions in the x - y plane with the x -axis score being the score on objective 1 and the y -axis score being the score on objective 2, the graph is called a **trade-off curve**. It is also called the **efficient frontier**. To illustrate, suppose that the set of feasible solutions for a multiple-objective problem is the shaded region bounded by the curve AB and the first quadrant in Figure 9.8. If we want to maximize both objectives 1 and 2, then the curve AB is the set of Pareto optimal points.

FIGURE 9.8
Trade-off Curve for
Maximizing Two
Objectives



As another illustration, suppose the set of feasible solutions for a multiple-objective problem is all shaded points in the first quadrant bounded from below by the curve AB in Figure 9.9. If our goal is to maximize objective 1 and minimize objective 2, then the curve AB is the set of Pareto optimal points.

FIGURE 9.9
Trade-off Curve for
Maximizing
Objective 1,
Minimizing
Objective 2



To find a trade-off curve, proceed according to the following steps.

Steps for Finding a Trade-off Curve

1. Choose an objective, say objective 1, and determine its best attainable value V_1 . For the solution attaining V_1 , find the value of objective 2 and label it V_2 . Then (V_1, V_2) is a point on the trade-off curve.
2. For values V of objective 2 that are better than V_2 , solve the optimization problem in step 1 with the additional constraint that the value of objective 2 is at least as good as V . Varying V (over values of V preferred to V_2) yields other points on the trade-off curve.

- In step 1 we obtained one “endpoint” of the trade-off curve. If we determine the best value of objective 2 that can be attained, we obtain the other endpoint of the trade-off curve.

We will illustrate the concept of Pareto optimality (and how to determine Pareto optimal solutions) with the following example.

EXAMPLE

9.2

MAXIMIZING PROFIT AND MINIMIZING POLLUTION AT CHEMCON

Chemcon plans to produce eight products. The profit per unit, the labor and raw material used per unit produced, and the pollution emitted per unit produced are given in Table 9.20. This table also includes lower and upper limits on production that Chemcon has imposed. Currently 1300 labor hours and 1000 units of raw material are available. Chemcon’s two objectives are to maximize profit and minimize pollution produced. Chemcon wants to graph the trade-off curve for this problem.

TABLE 9.20 Data for Chemcon Example

Product	1	2	3	4	5	6	7	8
Labor hrs/unit	5	5	1	4	3.5	4	2	3.5
Raw material/unit	3	4.5	5	5	4.5	2	3.5	3
Pollution/unit	25	29	35	26	17	25	28	6
Profit/unit	53	69	73	69	51	49	71	40
Min production	0	30	0	10	20	50	30	0
Max production	190	110	140	140	190	190	110	150

Solution

The model itself is a straightforward version of the product mix model from Chapter 3. We want the product mix that stays within the lower and upper production limits, uses no more labor or raw material than are available, keeps pollution low, and keeps profit high. None of the formulas in the spreadsheet model (see Figure 9.10 on page 466 and the file CHEMCON.XLS) presents anything new, so we will focus instead on the solution procedure.

Referring to the general three-step procedure for finding the trade-off curve, we let profit be objective 1 and pollution be objective 2. To obtain one endpoint of the curve (step 1), we maximize profit and *ignore* pollution. That is, we maximize the Profit cell and delete the constraint indicated in row 26 from the Solver dialog box. You can check that the solution has profit \$20,089 and pollution level 8980. (This is *not* the solution shown in the figure.) At the other end of the spectrum (step 3), we can minimize the pollution in cell B26 and ignore any constraint on profit. You can check that this solution has pollution level 3560 and profit \$8360. In other words, profit can get as high as \$20,089 by ignoring pollution or as low as \$8360 by focusing entirely on pollution, and pollution can get as low as 3560 by ignoring profit or as high as 8980 by focusing entirely on profit. These establish the extremes. Now we search for points in between (step 2).

Fortunately, SolverTable is just what we need. According to step 2, we need to constrain pollution to various degrees and see how large profit can be. This is indicated in Figure 9.10, where the objective is to maximize profit with an upper limit on pollution.

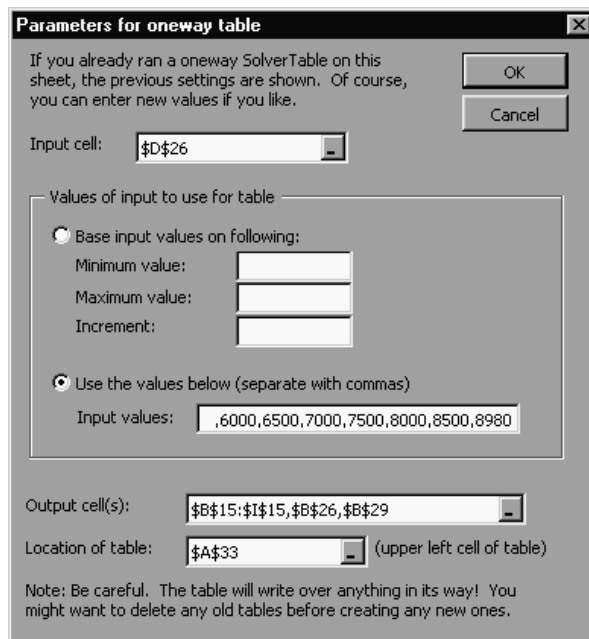
	A	B	C	D	E	F	G	H	I	J	K	L
1	Chemcon profit versus pollution model											
2												
3	Input data											
4	Product	1	2	3	4	5	6	7	8			
5	Labor hours/unit	5	5	1	4	3.5	4	2	3.5			
6	Raw material/unit	3	4.5	5	5	4.5	2	3.5	3			
7												
8	Pollution/unit	25	29	35	26	17	25	28	6			
9	Profit/unit	\$53	\$69	\$73	\$69	\$51	\$49	\$71	\$40			
10												
11	Production plan											
12	Product	1	2	3	4	5	6	7	8			
13	Min production	0	30	0	10	20	50	30	0			
14		<=	<=	<=	<=	<=	<=	<=	<=			
15	Units produced	0.0	30.0	0.0	10.0	21.1	50.0	48.6	150.0			
16		<=	<=	<=	<=	<=	<=	<=	<=			
17	Max production	190	110	140	140	190	190	110	150			
18												
19	Constraints on resources											
20		Used		Available								
21	Labor hours	1086.0	<=	1300								
22	Raw material	1000.0	<=	1000								
23												
24	Constraint on pollution											
25		Actual		Upper bound								
26		5000.0	<=	5000								
27												
28	Objective to maximize											
29	Profit	\$15,738										

Range names used:
UnitPollutions - B8:I8
UnitProfits - B9:I9
MinProduction - B13:I13
UnitsProduced - B15:I15
MaxProduction - B17:I17
Used - B21:B22
Available - D21:D22
Pollution - B26
UpBound - D26
Profit - B29

FIGURE 9.10 Chemcon Model

(We would get the same effect by minimizing pollution and putting a *lower* limit on profit.) The only upper limits on pollution we need to consider are those between the extremes, 3560 and 8980. Therefore, we use SolverTable with the setup shown in Figure 9.11. Note that we have used the option to enter non-equally-spaced inputs: 3560, 4000, 4500, and so on, ending with 8980. Alternatively, equally-spaced inputs could be used. All we require is a representative set of values between the extremes. The results appear in Figure 9.12.

FIGURE 9.11 SolverTable Dialog Box

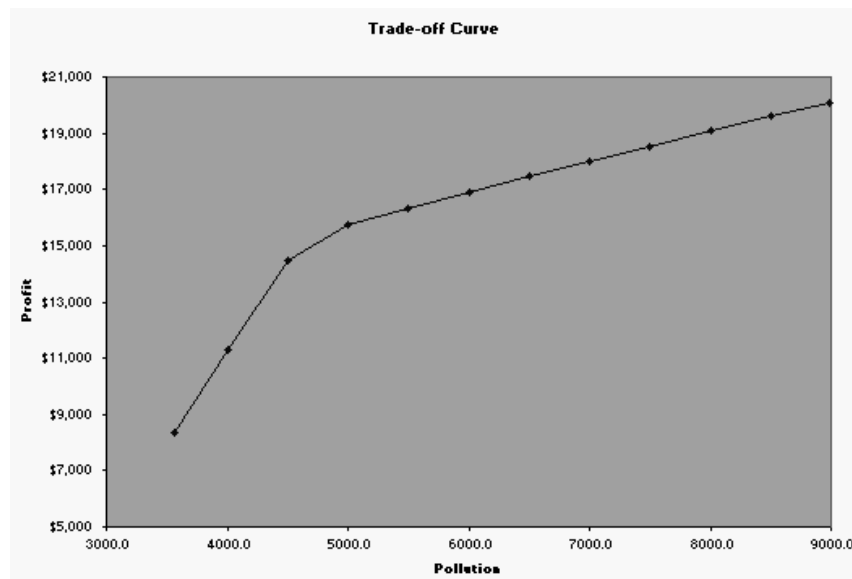


	A	B	C	D	E	F	G	H	I	J	K
31	Sensitivity of units produced, pollution, and profit to upper bound on pollution										
32	Upper bound	Product1	Product2	Product3	Product4	Product5	Product6	Product7	Product8	Pollution	Profit
33		\$B\$15	\$C\$15	\$D\$15	\$E\$15	\$F\$15	\$G\$15	\$H\$15	\$I\$15	\$B\$26	\$B\$29
34	3560	0.0	30.0	0.0	10.0	20.0	50.0	30.0	0.0	3560.0	\$8,360
35	4000	0.0	30.0	0.0	10.0	20.0	50.0	30.0	73.3	4000.0	\$11,293
36	4500	0.0	30.0	0.0	10.0	22.4	50.0	30.0	150.0	4500.0	\$14,480
37	5000	0.0	30.0	0.0	10.0	21.1	50.0	48.6	150.0	5000.0	\$15,738
38	5500	0.0	30.0	0.0	10.0	20.0	50.0	72.9	123.3	5500.0	\$16,336
39	6000	0.0	30.0	0.0	10.0	20.0	50.0	96.7	95.6	6000.0	\$16,916
40	6500	0.0	30.0	0.0	10.0	20.0	60.5	110.0	73.0	6500.0	\$17,474
41	7000	0.0	30.0	0.0	10.0	20.0	84.3	110.0	57.1	7000.0	\$18,006
42	7500	0.0	30.0	0.0	10.0	20.0	108.1	110.0	41.3	7500.0	\$18,537
43	8000	0.0	30.0	0.0	10.0	20.0	131.9	110.0	25.4	8000.0	\$19,069
44	8500	0.0	30.0	0.0	10.0	20.0	155.7	110.0	9.5	8500.0	\$19,601
45	8980	0.0	30.0	0.0	10.0	20.0	190.0	98.6	0.0	8980.0	\$20,089

FIGURE 9.12 SolverTable Results

These results show that as we allow more pollution, profit increases. Also, the product mix shifts considerably. Product 8, a low polluter with a low profit margin, eventually leaves the mix when pollution is allowed to increase, which makes sense. It is less clear why the level of product 6 increases so dramatically. It is only a moderate polluter and has a moderate profit margin, so the key is evidently that it requires low levels of labor and raw materials. The trade-off curve itself is created as an X-Y chart (with the points connected) directly from columns J and K of the table. It appears in Figure 9.13. This chart indicates that profit indeed increases as Chemcon allows more pollution, but at a decreasing rate. For example, when pollution is allowed to increase from 4000 to 4500, Chemcon can make an extra \$3187 in profit. However, when pollution is allowed to increase from 8000 to 8500, the extra profit is only \$532. All points below the curve are dominated—for a given level of pollution, the company can achieve a larger profit—and all points above the curve are unattainable.

FIGURE 9.13 Trade-off Curve for Profit Versus Pollution



Trade-off curves are not confined to linear models. The following example illustrates a trade-off curve in a situation where the objective is a nonlinear function of the changing cells.

TRADE-OFFS BETWEEN EXPOSURES TO MEN AND WOMEN AT LEON BURNIT

This example is a modification of the Burnit advertising example in Example 9.1. Now we assume that Burnit's client is concerned only with *two* groups of people, men and women. Also, the number of exposures to these groups is now a nonlinear square root function of the number of ads placed of any particular type. This implies a marginal decreasing effect of ads—each extra ad of a particular type reaches fewer extra people than the previous ad of this type. The data for this problem appear in Tables 9.21 and 9.22. The first of these specifies the proportionality constants for the square root exposure functions. For example, if 5 ads are placed in sports shows, this will achieve $15\sqrt{5} = 33.541$ million exposures to men, but only $5\sqrt{5} = 11.180$ million exposures to women. Evidently, what works well for men does not work so well for women, and vice versa. Given a budget of \$1.5 million, find the trade-off curve for exposures to men versus exposures to women.

TABLE 9.21 Proportionality Constants for Square Root Exposure Functions

	Sports Show	Game Show	News Show	Sitcom	Drama	Soap Opera
Men	15	3	7	7	8	1
Women	5	5	6	10	9	4

TABLE 9.22 Data on Ads for Burnit Example

	Sports Show	Game Show	News Show	Sitcom	Drama	Soap Opera
Cost/ad (\$1000s)	120	40	50	40	60	20
Lower limit	2	0	2	0	2	0
Upper limit	10	5	10	5	10	5

Solution

Again, the model itself is straightforward, as shown in Figure 9.14. (See the file BURNIT2.XLS.) We calculate the exposures achieved in rows 22 and 23 by entering the formula

=B8*SQRT(B\$17)

in cell B22 and copying it to the range B22:G23. We then sum these in the TotExp-ToWomen and TotExpToMen ranges, and we calculate the total cost in the usual way with the SUMPRODUCT function.

For the three-step trade-off curve procedure, we designate exposures to men as objective 1 and exposures to women as objective 2. For step 1, we maximize exposures to men and ignore women. That is, we do *not* include the constraint in row 30 in the Solver dialog box. You can check that the corresponding solution achieves 89.515 million exposures to men and 79.392 million exposures to women. Reversing the roles of men and women (step 3), you can check that if we maximize exposures to women and ignore men, we achieve 89.220 million exposures to women and only 84.899 million exposures to men.

	A	B	C	D	E	F	G	H	I	J	
1	Burnit nonlinear advertising model										
2											
3	Assumption: The number of exposures (in millions) to each group is proportional to the square root										
4	of the number of ads of a particular type shown.										
5											
6	Proportionality constants for exposure functions										
7		Sports ad	Game show ad	News show ad	Sitcom ad	Drama ad	Soap opera ads				
8	Exposures to men	15	3	7	7	8	1				
9	Exposures to women	5	5	6	10	9	4				
10											
11	Cost/ad (\$1,000s)	120	40	50	40	60	20				
12											
13	Advertising plan										
14		Sports ad	Game show ad	News show ad	Sitcom ad	Drama ad	Soap opera ad				
15	Minimum ads required	2	0	2	0	2	0				
16		<=	<=	<=	<=	<=	<=				
17	Number purchased	3,540	2,748	6,225	5,000	6,579	2,966				
18		<=	<=	<=	<=	<=	<=				
19	Maximum ads allowed	10	5	10	5	10	5				
20											
21	Exposures obtained										
22	Men	28,221	4,973	17,465	15,652	20,520	1,722				
23	Women	9,407	8,289	14,970	22,361	23,085	6,889				
24											
25	Budget constraint										
26	Total cost				Budget						
27		1500.000	<=	1500							
28	Constraint on minimal exposures to women										
29		Actual			Lower bound						
30		85.000	>=	85							
31											
32	Objective to maximize										
33	Exposures to men	88,554									

Range names used:
UnitCosts - B11:G11
MinAds - B15:G15
Ads - B17:G17
MaxAds - B19:G19
ExpToMen - B22:G22
ExpToWomen - B23:G23
TotCost - B26
Budget - D26
TotExpToWomen - B30
LowBound - D30
TotExpToMen - B33

FIGURE 9.14 Advertising Model

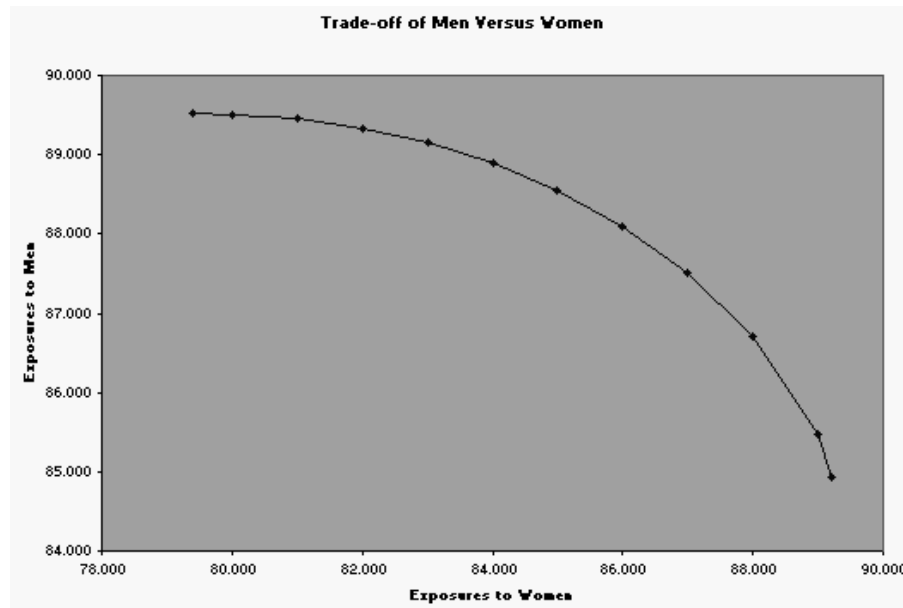
All other points on the trade-off curve are between these two extremes, and they can again be found easily with SolverTable. We now set up the Solver to maximize exposures to men, and we include the lower limit constraint on exposures to women in the Solver dialog box. (Do you see why it is a *lower* limit constraint in this example, whereas it was an upper limit constraint in the previous example? There we wanted to make pollution low. Here we want to make exposures to women high.) The lower limit cell (D30) becomes the single input cell for SolverTable, and we allow it to vary from (slightly greater than) 79.392 to (slightly less than) 89.220 with suitable values in between. The results appear in table form in Figure 9.15 and in graphical form in Figure 9.16 (page 470).

As we go down the table (or to the right in the chart), we require more exposures to women, which has an increasingly negative effect on exposures to men. Not surprisingly, the corresponding solutions place more ads in the shows watched predominantly

	A	B	C	D	E	F	G	H	I	
35	Sensitivity of ads purchased, exposures to women, and exposures to men to lower bound on exposures to women									
36		Sports ad	Game show ad	News show ad	Sitcom ad	Drama ad	Soap opera ad	To women	To men	
37		\$B\$17	\$C\$17	\$D\$17	\$E\$17	\$F\$17	\$G\$17	\$B\$30	\$B\$33	
38	79.393	4.833	1.744	6.072	5.000	5.508	0.776	79.393	89.515	
39	80	4.715	1.835	6.100	5.000	5.620	0.928	80.000	89.506	
40	81	4.503	1.994	6.143	5.000	5.807	1.215	81.000	89.449	
41	82	4.280	2.163	6.178	5.000	5.997	1.555	82.000	89.336	
42	83	4.048	2.347	6.204	5.000	6.186	1.954	83.000	89.156	
43	84	3.801	2.538	6.220	5.000	6.383	2.421	84.000	88.900	
44	85	3.540	2.745	6.228	5.000	6.578	2.969	85.000	88.554	
45	86	3.262	2.976	6.217	5.000	6.777	3.604	86.000	88.096	
46	87	2.964	3.225	6.189	5.000	6.979	4.357	87.000	87.500	
47	88	2.600	3.580	6.173	5.000	7.269	5.000	88.000	86.713	
48	89	2.057	4.276	6.207	5.000	7.863	5.000	89.000	85.478	
49	89.219	2.000	5.000	5.387	5.000	8.177	5.000	89.219	84.934	

FIGURE 9.15 SolverTable Results for Advertising Model

FIGURE 9.16
Trade-off Curve for
Advertising Example



by women (game shows, dramas, and soaps) and fewer ads in sports and news shows. The upper limit of 5 placed on sitcom ads prevents us from seeing how the number of sitcom ads would change if it were not constrained. It would probably change fairly dramatically, given that these ads are relatively cheap and they tend to reach more women than men.

Technical note We ran into two problems that you might experience. First, depending on the starting solution, it is possible that one of the changing cells can become slightly negative (due to numerical roundoff), in which case the SQRT function is undefined and you get an error message. To remedy this, add a constraint like $\text{Ads} \geq 0.0001$. Second, when we ran SolverTable, it indicated “no feasible solution” to the problem in row 49 of Figure 9.15, although we know there is a feasible solution. This can sometimes occur with nonlinear models, depending on the starting solution used. SolverTable uses the solution from the previous problem as the starting solution for the next problem. This seems reasonable, but it *can* produce this error. If it does, try running the Solver on this particular problem again with your own initial solution (such as all 0’s). This is what we did to get the values in row 49. ■

MODELING ISSUES

1. A trade-off curve is useful because it gives the ultimate decision maker many solutions to choose from, none of which is dominated by any others. However, it does *not* specify a “best” solution. The decision maker still has to make the difficult decision of which solution from the trade-off curve to implement. This can be done subjectively or with the help of a **multiattribute utility function**. However, it is difficult to estimate these types of functions, so their use in real-world applications has been limited.
2. We can generalize to a situation where there are more than two objectives by constructing trade-off curves between each *pair* of objectives. ■

PROBLEMS

Skill-Building Problems

19. Widgetco produces two types of widgets. Each widget is made of steel and aluminum and is assembled with skilled labor. The resources used and the per-unit profit contribution (ignoring cost of overtime labor purchased) for each type of widget are given in Table 9.23. At present, 200 pounds of steel, 300 pounds of aluminum, and 300 hours of labor are available. Extra overtime labor can be purchased for \$10 per hour. Construct a trade-off curve between the objectives of maximizing profit and minimizing overtime labor.

TABLE 9.23 Data for Widgetco Problem

	Type 1	Type 2
Steel	6 lb	12 lb
Aluminum	8 lb	20 lb
Skilled labor	11 hr	24 hr
Profit contribution	\$500	\$1100

20. Plantco produces three products. Three workers work for Plantco, and the company must determine which product(s) each worker should produce. The number of units each worker would produce if he or she spent the whole day producing each type of product is given in Table 9.24. The company is also interested in maximizing the happiness of its workers. The amount of happiness “earned” by a worker who spends the entire day producing a given product is given in Table 9.25. Construct a trade-off curve between the objectives of maximizing total units produced daily and total worker happiness.

TABLE 9.24 Worker Productivity for Plantco Problem

	Product 1	Product 2	Product 3
Worker 1	20	12	10
Worker 2	12	15	9
Worker 3	6	5	10

TABLE 9.25 Happiness Values for Plantco Problem

	Product 1	Product 2	Product 3
Worker 1	6	8	10
Worker 2	6	5	9
Worker 3	9	10	8

21. If a company spends a on advertising (measured in thousands of dollars) and charges a price of p dollars per unit, then it can sell $1000 - 100p + 20a^{1/2}$ units of the product. The cost per unit of producing the product is \$6. Construct a trade-off curve between the objectives of maximizing profit and maximizing the number of units sold.
22. GMCO produces three types of cars: compact, medium, and large. The variable cost per car and production capacity (per year) for each type of car are given in Table 9.26. The annual demand for each type of car depends on the prices of the three types of cars, given in Table 9.27. In this latter table, P_C is the price charged for a compact car (in thousands of dollars). The variables P_M and P_L are defined similarly for medium and large cars. Suppose that each compact gets 30 mpg, each medium car gets 25 mpg, and each large car gets 18 mpg. GMCO wants to keep the planet pollution free, so in addition to maximizing profit, it wants to maximize the average miles per gallon attained by the cars it sells. Construct a trade-off curve between these two objectives.

TABLE 9.26 Variable Costs and Production Capacities for GMCO Problem

	Cost	Production Capacity
Compact	\$10,000	2000
Medium	\$14,000	1500
Large	\$18,000	1000

TABLE 9.27 Demand Functions for GMCO Problem

	Demand
Compact	$2500 - 100P_C + 3P_M$
Medium	$1800 - 30P_M + 2P_C + P_L$
Large	$1300 - 20P_L + P_M$

23. In Section 5.5 of Chapter 5, we discussed crashing the length of the Franklin Company project. For this example construct a trade-off curve between the cost of crashing the project and duration of the project.
24. Referring to the Perlman & Brothers example from Chapter 7 (see Example 7.9), construct a trade-off curve between the chosen portfolio’s expected return and variance (with variance on the horizontal axis). In finance terminology this trade-off curve is called the **efficient frontier**.

9.4 THE ANALYTIC HIERARCHY PROCESS

When multiple objectives are important to a decision maker, it is often difficult to choose between alternatives. For example, if you are choosing a job, one job might offer the highest starting salary but rate poorly on other objectives such as quality of life in the city where the job is located and the nearness of the job to your family. Another job offer might rate highly on these latter objectives but have a relatively low starting salary. In this case it might be difficult for you to choose between job offers. Thomas Saaty's **Analytic Hierarchy Process (AHP)** provides a powerful tool that can be used to make decisions in situations where multiple objectives are present. We present an example to illustrate such a case.¹ (*Note:* We will use matrix notation and matrix multiplication in this section. You may need to review the discussion of matrices in Section 7.9.)

EXAMPLE

9.4

USING AHP TO SELECT A JOB

Jane is about to graduate from college and is trying to determine which job to accept. She plans to choose among the offers by determining how well each job offer meets the following four objectives:

- Objective 1: High starting salary
- Objective 2: Quality of life in city where job is located
- Objective 3: Interest of work
- Objective 4: Nearness of job to family

Solution

To illustrate how AHP works, suppose that Jane is facing three job offers and must determine which offer to accept. In this example there are four objectives, as listed above. For each objective, AHP generates a weight (by a method to be described shortly). By convention, the weights are always chosen so that they sum to 1. Suppose that Jane's weights are $w_1 = 0.5115$, $w_2 = 0.0986$, $w_3 = 0.2433$, and $w_4 = 0.1466$. These weights indicate that a high starting salary is the most important objective, followed by interest of work, nearness to family, and quality of life.

Next suppose that Jane determines (again by a method that will be described shortly) how well each job "scores" on each objective. For example, suppose these scores are those listed in Table 9.28. We see from this table that job 1 best meets the

TABLE 9.28 Job Scores on Objectives in AHP Example

	Salary	Quality of Life	Interest of Work	Nearness to Family
Job 1	0.5714	0.1593	0.0882	0.0824
Job 2	0.2857	0.2519	0.6687	0.3151
Job 3	0.1429	0.5889	0.2431	0.6025

¹The leading software package for implementing AHP is Expert Choice, developed by ExpertTM Choice Inc.

objective of a high starting salary, but scores worst on all other objectives. Note that the scores of the jobs on each objective are “normalized,” which means that for each objective, the sum of the scores of the jobs on that objective is 1.

Given the weights for the objectives and the scores shown in Table 9.28, Jane can now determine which job offer to accept. Specifically, for each job we calculate an overall score that is a weighted sum of the scores for that job, using the w 's as weights. For example, the overall score for job 1 weights the scores in the first row of Table 9.28:

$$\begin{aligned} \text{Job 1 score} &= 0.5115(0.5714) + 0.0986(0.1593) \\ &\quad + 0.2433(0.0882) + 0.1466(0.0824) \\ &= 0.3415 \end{aligned}$$

Similarly, the overall scores for jobs 2 and 3 are obtained by weighting the scores in the second and third rows of Table 9.28:

$$\begin{aligned} \text{Job 2 score} &= 0.5115(0.2857) + 0.0986(0.2519) \\ &\quad + 0.2433(0.6687) + 0.1466(0.3151) \\ &= 0.3799 \end{aligned}$$

$$\begin{aligned} \text{Job 3 score} &= 0.5115(0.1429) + 0.0986(0.5889) \\ &\quad + 0.2433(0.2431) + 0.1466(0.6025) \\ &= 0.2786 \end{aligned}$$

Because the overall score for job 2 is the largest, AHP suggests that Jane should accept this job.

Pairwise Comparison Matrices To obtain the weights for the various objectives, we begin by forming a matrix A , known as the pairwise comparison matrix. The entry in row i and column j of A , labeled a_{ij} , indicates how much more (or less) important objective i is than objective j . “Importance” is measured on an integer-valued scale from 1 to 9, with each number having the interpretation shown in Table 9.29. The phrases in this table, such as “strongly more important than,” are suggestive only. They simply indicate discrete points on a continuous scale that can be used to compare the relative importance of any two objectives.

TABLE 9.29 Interpretation of Values in Pairwise Comparison Matrix

Value of a_{ij}	Interpretation
1	Objectives i and j are equally important.
3	Objective i is slightly more important than j .
5	Objective i is strongly more important than j .
7	Objective i is very strongly more important than j .
9	Objective i is absolutely more important than j .

For example, if $a_{13} = 3$, then objective 1 is slightly more important than objective 3. If $a_{ij} = 4$, a value not in the table, then objective i is somewhere between slightly and strongly more important than objective j . If objective i is *less* important than objective j , we use the reciprocal of the appropriate index. For example, if objective i is slightly less important than objective j , then $a_{ij} = 1/3$. Finally, for all objectives i , we use the convention that $a_{ii} = 1$.

For consistency, it is necessary to set $a_{ji} = 1/a_{ij}$. For example, if $a_{13} = 3$, then it is necessary to have $a_{31} = 1/3$. This simply states that if objective 1 is slightly more important than objective 3, then objective 3 is slightly less important than job 1. It is usually easier to determine all a_{ij} 's that are greater than 1 and then use the relationship $a_{ji} = 1/a_{ij}$ to determine the remaining entries in the pairwise comparison matrix.

To illustrate, suppose that Jane has identified the following pairwise comparison matrix for her four objectives:

$$A = \begin{bmatrix} 1 & 5 & 2 & 4 \\ 1/5 & 1 & 1/2 & 1/2 \\ 1/2 & 2 & 1 & 2 \\ 1/4 & 2 & 1/2 & 1 \end{bmatrix}$$

The rows and columns of A each correspond to Jane's four objectives: salary, quality of life, interest of work, and nearness to family. Considering the first row, for example, she believes that salary is more important, in various degrees, than quality of life, interest of work, and nearness to family.

The entries in this matrix have built-in pairwise consistency because we require $a_{ij} = 1/a_{ji}$ for each i and j . However, they might not be consistent when three (or more) alternatives are considered simultaneously. For example, Jane claims that salary is strongly more important than quality of life ($a_{12} = 5$) and that salary is very slightly more important than interesting work ($a_{13} = 2$). But she also says that interesting work is very slightly more important than quality of life ($a_{32} = 2$). The question is whether these ratings are all consistent with one another. They are not, at least not exactly. It can be shown that some of Jane's pairwise comparisons are slightly inconsistent. When a person is asked to make this many pairwise comparisons, slight inconsistencies are common and fortunately do not cause serious difficulties. An index that can be used to measure the consistency of Jane's preferences will be discussed later in this section.

Determining the Weights Although the ideas behind AHP are fairly intuitive, the mathematical reasoning required to derive the weights for the objectives is quite advanced. Therefore, we simply describe how it is done.

Starting with the pairwise comparison matrix A , we find the weights for Jane's four objectives using the following two steps.

1. For each of the columns of A , divide each entry in the column by the sum of the entries in the column. This yields a new matrix (call it A_{norm} , for "normalized") in which the sum of the entries in each column is 1.

For Jane's pairwise comparison matrix, this step yields

$$A_{\text{norm}} = \begin{bmatrix} 0.5128 & 0.5000 & 0.5000 & 0.5333 \\ 0.1026 & 0.1000 & 0.1250 & 0.0667 \\ 0.2564 & 0.2000 & 0.2500 & 0.2667 \\ 0.1282 & 0.2000 & 0.1250 & 0.1333 \end{bmatrix}$$

2. Estimate w_i , the weight for objective i , as the average of the entries in row i of A_{norm} .

For Jane's matrix this yields

$$w_1 = \frac{0.5128 + 0.5000 + 0.5000 + 0.5333}{4} = 0.5115$$

$$w_2 = \frac{0.1026 + 0.1000 + 0.1250 + 0.0667}{4} = 0.0986$$

$$w_3 = \frac{0.2564 + 0.2000 + 0.2500 + 0.2667}{4} = 0.2433$$

$$w_4 = \frac{0.1282 + 0.2000 + 0.1250 + 0.1333}{4} = 0.1466$$

Intuitively, why does w_1 approximate the weight for objective 1 (salary)? Here is the reasoning. The proportion of weight that salary is given in pairwise comparisons of each objective to salary is 0.5128. Similarly, 0.50 represents the proportion of total weight that salary is given in pairwise comparisons of each objective to quality of life. Therefore, we see that each of the four numbers averaged to obtain w_1 represents a measure of the total weight attached to salary. Averaging these numbers should give a good estimate of the proportion of the total weight given to salary.

Determining the Score of Each Decision Alternative on Each Objective Now that we have determined the weights, we need to determine how well each job scores on each objective. To determine these scores, we use the same scale described in Table 9.29 to construct a pairwise comparison matrix for each objective. Consider the salary objective, for example. Suppose that Jane assesses the following pairwise comparison matrix. We denote this matrix as A_1 because it reflects her comparisons of the three jobs with respect to the first objective, salary.

$$A_1 = \begin{bmatrix} 1 & 2 & 4 \\ 1/2 & 1 & 2 \\ 1/4 & 1/2 & 1 \end{bmatrix}$$

The rows and columns of this matrix correspond to the three jobs. For example, the first row means that Jane believes job 1 is superior to job 2 (and even more superior to job 3) in terms of salary. To find the relative scores of the three jobs on salary, we now apply the *same* two-step procedure as above to the salary pairwise comparison matrix A_1 . That is, we first divide each column entry by the column sum to obtain

$$A_{1,\text{norm}} = \begin{bmatrix} 0.5714 & 0.5714 & 0.5714 \\ 0.2857 & 0.2857 & 0.2857 \\ 0.1429 & 0.1429 & 0.1429 \end{bmatrix}$$

Then we average the numbers in each row to obtain the vector of scores for the three jobs on salary, denoted by S_1 :

$$S_1 = \begin{bmatrix} 0.5714 \\ 0.2857 \\ 0.1429 \end{bmatrix}$$

That is, the scores for jobs 1, 2, and 3 on salary are 0.5714, 0.2857, and 0.1429. In terms of salary, job 1 is clearly the favorite.

Next, we repeat these calculations for Jane's other objectives. Each of these objectives requires a pairwise comparison matrix, which we will denote as A_2 , A_3 , and A_4 . Suppose that Jane's pairwise comparison matrix for quality of life is

$$A_2 = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 2 & 1 & 1/3 \\ 3 & 3 & 1 \end{bmatrix}$$

Then the corresponding normalized matrix is

$$A_{2,\text{norm}} = \begin{bmatrix} 0.1667 & 0.1111 & 0.2000 \\ 0.3333 & 0.2222 & 0.2000 \\ 0.5000 & 0.6667 & 0.6000 \end{bmatrix}$$

and by averaging, we obtain

$$S_2 = \begin{bmatrix} 0.1593 \\ 0.2519 \\ 0.5889 \end{bmatrix}$$

Here, job 3 is the clear favorite. However, this might not have much impact because Jane puts relatively little weight on quality of life.

For interest of work, suppose the pairwise comparison matrix is

$$A_3 = \begin{bmatrix} 1 & 1/7 & 1/3 \\ 7 & 1 & 3 \\ 3 & 1/3 & 1 \end{bmatrix}$$

Then the same types of calculations show that the scores for jobs 1, 2, and 3 on interest of work are

$$S_3 = \begin{bmatrix} 0.0882 \\ 0.6687 \\ 0.2431 \end{bmatrix}$$

Finally, suppose the pairwise comparison matrix for nearness to family is

$$A_4 = \begin{bmatrix} 1 & 1/4 & 1/7 \\ 4 & 1 & 1/2 \\ 7 & 2 & 1 \end{bmatrix}$$

In this case the scores for jobs 1, 2, and 3 on nearness to family are

$$S_4 = \begin{bmatrix} 0.0824 \\ 0.3151 \\ 0.6025 \end{bmatrix}$$

Determining the Best Alternative Let's summarize what we have determined so far. Jane first assesses a pairwise comparison matrix A that measures the relative importance of each of her objectives to one another. From this matrix we obtain a vector of weights that summarizes the relative importance of the objectives. Next, Jane assesses a pairwise comparison matrix A_i for each objective i . This matrix measures how well each job compares to other jobs with regard to this objective. For each matrix A_i we obtain a vector of scores S_i that summarizes how the jobs compare in terms of achieving objective i .

The final step is to combine the scores in the S_i vectors with the weights in the w vector. Actually, we have already done this. Note that the columns of Table 9.28 are the S_i vectors we just obtained. If we form a matrix S of these score vectors and multiply this matrix by w , we obtain a vector of overall scores for each job, as shown below:

$$Sw = \begin{bmatrix} 0.5714 & 0.1593 & 0.0882 & 0.0824 \\ 0.2857 & 0.2519 & 0.6687 & 0.3151 \\ 0.1429 & 0.5889 & 0.2431 & 0.6025 \end{bmatrix} \times \begin{bmatrix} 0.5115 \\ 0.0986 \\ 0.2433 \\ 0.1466 \end{bmatrix} = \begin{bmatrix} 0.3415 \\ 0.3799 \\ 0.2786 \end{bmatrix}$$

These are the same overall scores that we obtained earlier. As before, the largest of these overall scores is for job 2, so AHP suggests that Jane should accept this job. Job 1 follows closely behind, with job 3 somewhat farther behind.

Checking for Consistency As mentioned earlier, any pairwise comparison matrix can suffer from inconsistencies. We now describe a procedure to check for inconsistencies.

We illustrate this on the A matrix and its associated vector of weights w . The same procedure can be used on any of the A_i matrices and their associated “weights” vector S_i .

1. Compute Aw . For the example, we obtain

$$Aw = \begin{bmatrix} 1 & 5 & 2 & 4 \\ 1/5 & 1 & 1/2 & 1/2 \\ 1/2 & 2 & 1 & 2 \\ 1/4 & 2 & 1/2 & 1 \end{bmatrix} \times \begin{bmatrix} 0.5115 \\ 0.0986 \\ 0.2433 \\ 0.1466 \end{bmatrix} = \begin{bmatrix} 2.0774 \\ 0.3958 \\ 0.9894 \\ 0.5933 \end{bmatrix}$$

2. Find the ratio of each element of Aw to the corresponding weight in w and average these ratios. For the example, this calculation is

$$\frac{\frac{2.0774}{0.5115} + \frac{0.3958}{0.0986} + \frac{0.9894}{0.2433} + \frac{0.5933}{0.1466}}{4} = 4.0477$$

3. Compute the consistency index (labeled CI) as

$$CI = \frac{(\text{Step 2 result}) - n}{n - 1}$$

where n is the number of objectives. For the example this is $CI = \frac{4.0477-4}{4-1} = 0.0159$.

4. Compare CI to the random index (labeled RI) in Table 9.30 for the appropriate value of n .

TABLE 9.30 Random Indices for Consistency Check for AHP Example

n	2	3	4	5	6	7	8	9	10
RI	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.51

To be a perfectly consistent decision maker, each ratio in step 2 should equal n . This implies that a perfectly consistent decision maker has $CI = 0$. The values of RI in Table 9.30 give the average value of CI if the entries in A were chosen at random (subject to the constraints that a_{ii} ’s must equal 1, and $a_{ij} = 1/a_{ji}$). If the ratio of CI to RI is sufficiently small, then the decision maker’s comparisons are probably consistent enough to be useful. Saaty suggests that if $CI/RI < 0.10$, then the degree of consistency is satisfactory, whereas if $CI/RI > 0.10$, serious inconsistencies exist and AHP may not yield meaningful results. In Jane’s example, $CI/RI = 0.0159/0.90 = 0.0177$, which is much less than 0.10. Therefore, Jane’s pairwise comparison matrix A does not exhibit any serious inconsistencies. (You can check that the same is true of her other pairwise comparison matrices A_1 through A_4 .)



DEVELOPING THE SPREADSHEET MODEL

We now show how to implement AHP on a spreadsheet. (See Figure 9.17 on page 478 and the file JOBSAHP.XLS.)

- 1 **Inputs.** Enter the pairwise comparison matrices in the shaded ranges. (Note that you can enter fractions such as $1/7$ in cell C24, and have them appear as fractions, by formatting the cells with the Fraction option.)
- 2 **Normalized matrix.** Calculate the normalized matrix for the first pairwise comparison matrix in the range G5:J8. This can be done quickly as follows. Starting with

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	Job selection using analytical hierarchy process												
2													
3	Pairwise comparisons among objectives					Normalized matrix					Weights		
4		Salary	Life quality	Work interest	Near family								
5	Salary	1	5	2	4		0.5128	0.5000	0.5000	0.5333		0.5115	
6	Life quality	1/5	1	1/2	1/2		0.1028	0.1000	0.1250	0.0667		0.0986	
7	Work interest	1/2	2	1	2		0.2564	0.2000	0.2500	0.2667		0.2433	
8	Near family	1/4	2	1/2	1		0.1282	0.2000	0.1250	0.1333		0.1466	
9													
10	Pairwise comparisons among jobs on salary					Normalized matrix					Scores		
11		Job 1	Job 2	Job 3									
12	Job 1	1	2	4			0.5714	0.5714	0.5714			0.5714	
13	Job 2	1/2	1	2			0.2857	0.2857	0.2857			0.2857	
14	Job 3	1/4	1/2	1			0.1429	0.1429	0.1429			0.1429	
15													
16	Pairwise comparisons among jobs on quality of life					Normalized matrix					Scores		
17		Job 1	Job 2	Job 3									
18	Job 1	1	1/2	1/3			0.1667	0.1111	0.2000			0.1593	
19	Job 2	2	1	1/3			0.3333	0.2222	0.2000			0.2519	
20	Job 3	3	3	1			0.5000	0.6667	0.6000			0.5889	
21													
22	Pairwise comparisons among jobs on interest of work					Normalized matrix					Scores		
23		Job 1	Job 2	Job 3									
24	Job 1	1	1/7	1/3			0.0909	0.0968	0.0769			0.0882	
25	Job 2	7	1	3			0.6364	0.6774	0.6923			0.6687	
26	Job 3	3	1/3	1			0.2727	0.2258	0.2308			0.2431	
27													
28	Pairwise comparisons among jobs on nearness to family					Normalized matrix					Scores		
29		Job 1	Job 2	Job 3									
30	Job 1	1	1/4	1/7			0.0833	0.0769	0.0870			0.0824	
31	Job 2	4	1	1/2			0.3333	0.3077	0.3043			0.3151	
32	Job 3	7	2	1			0.5833	0.6154	0.6087			0.6025	
33													
34	Determining best job												
35	Matrix of scores					Weighted scores					Range names used: Scores - B37:E39 Weights - L5:L8		
36		Salary	Life quality	Work interest	Near family								
37	Job 1	0.5714	0.159	0.088	0.082		0.3415						
38	Job 2	0.2857	0.252	0.669	0.315		0.3799						
39	Job 3	0.1429	0.589	0.243	0.602		0.2786						

FIGURE 9.17 AHP Job Selection Model

the cursor in cell G5, drag the cursor so that the range G5:J8 is highlighted. Then type the formula

=B5/SUM(B\$5:B\$8)

and press Control-Enter (both keys at once). You have just learned a quicker way than copying and pasting!

3 Weights of objectives. In the range L5:L8, calculate the weights for each objective. Again, do this the quick way. Starting with the cursor in cell L5, highlight the range L5:L8. Then type the formula

=AVERAGE(G5:J5)

and press Control-Enter.

4 Scores for jobs on objectives. Repeat the same calculations in steps 2 and 3 for the other pairwise comparison matrices to obtain the normalized matrices in columns G through I and scores vectors in column L.

5 Overall job scores. In the Scores range form a matrix of job scores on the various objectives. To get the score vector in the range L12:L14 into the Scores range, for example, highlight the Scores range, type the formula

=L12

and press Control-Enter. Do likewise for the other three scores vectors in column L.

Then to obtain the overall job scores (from the formula Sw), highlight the range G37:G39, type the formula

=MMULT(Scores,Weights)

and press Control-Shift-Enter. (Remember that Control-Shift-Enter is used to enter a matrix function. In contrast, Control-Enter is equivalent to copying a formula to a highlighted range.)

Again we see that job 2 is the most preferred of the three jobs.

Calculating the Consistency Index We now show how to compute the consistency index CI for each of the pairwise comparison matrices. (See Figure 9.18, which is also part of the file JOBSAHP.XLS. Note that we have hidden columns G through K to save space. These contain the normalized matrices from step 2 above.) The following steps are relevant for the first pairwise comparison matrix. The others are done in analogous fashion.

1 Product of comparison matrix and vector of weights (or scores). Calculate the product of the first pairwise comparison matrix and the weights vector in the range N5:N8 by highlighting this range, typing

=MMULT(B5:E8,L5:L8)

and pressing Control-Shift-Enter.

	A	B	C	D	E	F	L	M	N	O	
1	Job selection using analytical hierarchy process								Checking consistency		
2											
3	Pairwise comparisons among objectives						Weights		Product	Ratios	
4		Salary	Life quality	Work interest	Near family						
5	Salary	1	5	2	4		0.5115		2.0774	4.0611	
6	Life quality	1/5	1	1/2	1/2		0.0986		0.3958	4.0161	
7	Work interest	1/2	2	1	2		0.2433		0.9894	4.0672	
8	Near family	1/4	2	1/2	1		0.1466		0.5933	4.0459	
9									CI	0.0159	
10	Pairwise comparisons among jobs on salary						Scores		CI/RI	0.0176	
11		Job 1	Job 2	Job 3							
12	Job 1	1	2	4			0.5714		1.7143	3	
13	Job 2	1/2	1	2			0.2857		0.8571	3	
14	Job 3	1/4	1/2	1			0.1429		0.4286	3	
15									CI	0	
16	Pairwise comparisons among jobs on quality of life						Scores		CI/RI	0.0000	
17		Job 1	Job 2	Job 3							
18	Job 1	1	1/2	1/3			0.1593		0.4815	3.0233	
19	Job 2	2	1	1/3			0.2519		0.7667	3.0441	
20	Job 3	3	3	1			0.5889		1.8222	3.0943	
21									CI	0.0270	
22	Pairwise comparisons among jobs on interest of work						Scores		CI/RI	0.0465	
23		Job 1	Job 2	Job 3							
24	Job 1	1	1/7	1/3			0.0882		0.2648	3.0018	
25	Job 2	7	1	3			0.6687		2.0154	3.0139	
26	Job 3	3	1/3	1			0.2431		0.7306	3.0054	
27									CI	0.0035	
28	Pairwise comparisons among jobs on nearness to family						Scores		CI/RI	0.0061	
29		Job 1	Job 2	Job 3							
30	Job 1	1	1/4	1/7			0.0824		0.2473	3.0005	
31	Job 2	4	1	1/2			0.3151		0.9460	3.0019	
32	Job 3	7	2	1			0.6025		1.8096	3.0035	
33									CI	0.0010	
34									CI/RI	0.0017	

FIGURE 9.18 Checking for Consistency

2 Ratios. In cell O5 calculate the ratio of the two cells to its left with the formula `=N5/L5`

and copy this to the range O6:O8.

3 Consistency index. Calculate the consistency index *CI* in cell O9 with the formula

`=(AVERAGE(O5:O8)-4)/3`

Then in cell O10 calculate the ratio of *CI* to *RI* (for $n = 4$) with the formula

`=O9/0.90`

(The 0.90 comes from Table 9.30. For the other four pairwise comparison matrices in Figure 9.18, we use $n = 3$ and $RI = 0.58$.)

As Figure 9.18 illustrates, all of the pairwise comparison matrices are quite consistent—the *CI/RI* ratio for each is well less than 0.10. ■

MODELING ISSUES

1. In Jane's job selection example, suppose that quality of life depends on two "subobjectives": recreational and educational facilities. Then we need a pairwise comparison matrix to calculate the proportion of the quality of life score that is determined by recreational facilities and the proportion that is determined by educational facilities. Next, we need to determine how each job scores (separately) on recreational facilities and educational facilities. Then we can again determine a "quality of life score" for each job and proceed with AHP as before. Using this idea, AHP can handle a *hierarchy* of objectives and subobjectives—hence the term "hierarchy" in the name of the procedure. (See Problem 31.)
2. Although the JOBSAHP.XLS file can be used as a "template" for other AHP problems, it is clear by now that typical users would not want to go to all of this trouble to create a spreadsheet model, certainly not from scratch. If you intend to make any real decisions with AHP, you will want to acquire special-purpose software such as Expert Choice. To give you some feeling for this software, we have included a VBA application of AHP in the companion book *VBA for Modelers* (which can be purchased separately). ■

ADDITIONAL APPLICATIONS

Automated Manufacturing Decisions Using AHP

Weber (1993) reports the successful use of AHP in deciding which of several technologies to purchase for automated manufacturing. As he discusses, these decisions can have several types of impacts: quantitative financial (such as purchase cost), quantitative nonfinancial (such as throughput, cycle time, and scrap, which are difficult to translate directly into dollars), and qualitative (such as product quality and manufacturing flexibility, which are also difficult to translate into dollars). When the decision maker is trying to rate the different technologies along nonmonetary criteria, then he or she should use the method discussed in this section. (For example, how much more do you prefer technology 1 to technology 2 in the area of product quality?) However, he

advises that when quantitative financial data are available (for example, technology 1 costs twice as much as technology 2), then this “objective” information should be used in the AHP preference matrices. Weber developed a software package called AutoMan to implement the AHP method. This software has been purchased by over 800 decision makers since its first release in 1989.

Other Applications of AHP

We close by noting that AHP has been used by decision makers in many areas, including accounting, finance, marketing, energy resource planning, microcomputer selection, sociology, architecture, and political science. See Zahedi (1986), Golden et al. (1989), and Saaty (1988) for a discussion of applications of AHP. ■

PROBLEMS

Skill-Building Problems

25. Each professor’s annual salary increase is determined by his or her performance in three areas: teaching, research, and service to the university. The administration has assessed the pairwise comparison matrix for these objectives as shown in Table 9.31. The administration has compared two professors with regard to their teaching, research, and service over the past year. The pairwise comparison matrices are shown in Table 9.32.

	Teaching	Research	Service
Teaching	1	1/3	5
Research	3	1	7
Service	1/5	1/7	1

- Which professor should receive a bigger raise?
- Does AHP indicate how large a raise each professor should be given?
- Check the pairwise comparison matrix for consistency.

TABLE 9.32 Pairwise Comparisons of Professors in Salary Problem

	Teaching	
	Professor 1	Professor 2
Professor 1	1	4
Professor 2	1/4	1

	Research	
	Professor 1	Professor 2
Professor 1	1	1/3
Professor 2	3	1

	Service	
	Professor 1	Professor 2
Professor 1	1	6
Professor 2	1/6	1

26. Your company is about to purchase a new personal computer. Three objectives are important in determining which computer you should purchase: cost, user-friendliness, and software availability. The pairwise comparison matrix for these objectives is shown in Table 9.33. Three computers are being

TABLE 9.33 Pairwise Comparisons of Objectives in Computer Purchasing Problem

	Cost	User-friendliness	Software Availability
Cost	1	1/4	1/5
User-friendliness	4	1	1/2
Software availability	5	2	1

considered for purchase. The performance of each computer with regard to each objective is indicated by the pairwise comparison matrices in Table 9.34.

- a. Which computer should you purchase?
 - b. Check the pairwise comparison matrices for consistency.
27. You are ready to select your mate for life and have determined that physical attractiveness, intelligence, and personality are key factors in selecting a satisfactory mate. Your pairwise comparison matrix for these objectives is shown in Table 9.35. Three people (Chris, Jamie, and Pat) are begging to be your mate. (This problem attempts to be gender-neutral!) Your view of these people's attractiveness, intelligence, and personality is given in the pairwise comparison matrices in Table 9.36.
- a. Who should you choose as your lifetime mate?
 - b. Evaluate all pairwise comparison matrices for consistency.
28. In determining where to invest your money, two objectives, expected rate of return and degree of risk, are considered to be equally important. Two investments (1 and 2) have the pairwise comparison matrices shown in Table 9.37.
- a. How would you rank these investments?
 - b. Now suppose another investment (investment 3) is available. The pairwise comparison

TABLE 9.36 Pairwise Comparisons of People in Mating Problem

Attractiveness	Chris	Jamie	Pat
Chris	1	5	3
Jamie	1/5	1	1/2
Pat	1/3	2	1
Intelligence	Chris	Jamie	Pat
Chris	1	1/6	1/4
Jamie	6	1	2
Pat	4	1/2	1
Personality	Chris	Jamie	Pat
Chris	1	4	1/4
Jamie	1/4	1	1/9
Pat	4	9	1

matrices for these investments are shown in Table 9.38. (Observe that the entries in the comparison matrices for investments 1 and 2 have not changed.) How would you now rank the investments? Contrast your ranking of investments 1 and 2 with your answer from part a.

TABLE 9.34 Pairwise Comparisons of Computers in Computer Purchasing Problem

Cost	Computer 1	Computer 2	Computer 3
Computer 1	1	3	5
Computer 2	1/3	1	2
Computer 3	1/5	1/2	1
User-friendliness	Computer 1	Computer 2	Computer 3
Computer 1	1	1/3	1/2
Computer 2	3	1	5
Computer 3	2	1/5	1
Software Availability	Computer 1	Computer 2	Computer 3
Computer 1	1	1/2	1/7
Computer 2	2	1	1/5
Computer 3	7	5	1

TABLE 9.35 Pairwise Comparisons of Objectives in Mating Problem

	Attractiveness	Intelligence	Personality
Attractiveness	1	3	5
Intelligence	1/3	1	3
Personality	1/5	1/3	1

TABLE 9.37 Pairwise Comparisons in Investment Problem

Expected Return	Investment 1	Investment 2
Investment 1	1	1/2
Investment 2	2	1
Degree of Risk	Investment 1	Investment 2
Investment 1	1	3
Investment 2	1/3	1

TABLE 9.38 Expanded Pairwise Comparisons in Investment Problem

Expected Return	Investment 1	Investment 2	Investment 3
Investment 1	1	1/2	4
Investment 2	2	1	8
Investment 3	1/4	1/8	1
Degree of Risk	Investment 1	Investment 2	Investment 3
Investment 1	1	3	1/2
Investment 2	1/3	1	1/6
Investment 3	2	6	1

29. You are trying to determine which MBA program to attend. You have been accepted at two schools: Indiana and Northwestern. You have chosen three attributes to use in helping you make your decision: cost, starting salary for graduates, and ambience of school (can we party there?). Your pairwise comparison matrix for these attributes is shown in Table 9.39. For each attribute, the pairwise comparison matrix for Indiana and Northwestern is shown in Table 9.40. Which MBA program should you attend?

TABLE 9.39 Pairwise Comparisons of Attributes in School Problem

	Cost	Salary	Ambience
Cost	1	1/4	2
Salary	4	1	7
Ambience	1/2	1/7	1

30. You are trying to determine which of two secretarial candidates (John or Sharon) to hire. The three objectives that are important to your decision are personality, typing ability, and intelligence. You have assessed the pairwise comparison matrix for the three objectives in Table 9.41. The score of each employee on each objective is shown in

TABLE 9.40 Pairwise Comparisons of Indiana and Northwestern

Cost	Indiana	Northwestern
Indiana	1	6
Northwestern	1/6	1
Salary	Indiana	Northwestern
Indiana	1	1/6
Northwestern	6	1
Ambience	Indiana	Northwestern
Indiana	1	4
Northwestern	1/4	1

TABLE 9.41 Pairwise Comparisons of Objectives in Secretary Problem

	Personality	Typing Ability	Intelligence
Personality	1	1/4	1/3
Typing ability	4	1	1/2
Intelligence	3	2	1

Table 9.42 (page 484). If you follow the AHP method, which employee should you hire?

TABLE 9.42 Candidate Scores in Secretary Problem

	Personality	Typing Ability	Intelligence
John	0.4	0.6	0.2
Sharon	0.6	0.4	0.8

Skill-Extending Problems

31. A consumer is trying to determine which type of frozen dinner to eat. She considers three attributes to be important: taste, nutritional value, and price. Nutritional value is considered to be determined by cholesterol and sodium level. Three types of dinners are under consideration. The pairwise comparison matrix for the three attributes is shown in Table 9.43. Among the three frozen dinners, the pairwise comparison matrix for each attribute is shown in Table 9.44. To determine how each dinner rates on nutrition, you will need the pairwise comparison matrix for cholesterol and sodium in Table 9.45.

TABLE 9.43 Pairwise Comparisons of Attributes for Food Problem

	Taste	Nutrition	Price
Taste	1	3	1/2
Nutrition	1/3	1	1/5
Price	2	5	1

TABLE 9.44 Pairwise Comparisons of Dinners in Food Problem

Taste	Dinner 1	Dinner 2	Dinner 3
Dinner 1	1	5	3
Dinner 2	1/5	1	1/2
Dinner 3	1/3	2	1
Sodium	Dinner 1	Dinner 2	Dinner 3
Dinner 1	1	1/7	1/3
Dinner 2	7	1	2
Dinner 3	3	1/2	1
Cholesterol	Dinner 1	Dinner 2	Dinner 3
Dinner 1	1	1/8	1/4
Dinner 2	8	1	2
Dinner 3	4	1/2	1
Price	Dinner 1	Dinner 2	Dinner 3
Dinner 1	1	4	1/2
Dinner 2	1/4	1	1/6
Dinner 3	2	6	1

TABLE 9.45 Pairwise Comparison of Sodium and Cholesterol in Food Problem

	Cholesterol	Sodium
Cholesterol	1	5
Sodium	1/5	1

Which frozen dinner would the consumer prefer? (*Hint:* The nutrition score for a dinner equals the score of the dinner on sodium multiplied by the weight for sodium plus the score for the dinner on cholesterol multiplied by the weight for cholesterol.)

32. Based on Lin et al. (1984). You have been hired by Arthur Ross to determine which of the following accounts receivable methods should be used in an audit of the Keating Five and Dime Store: analytic review (method 1), confirmations (method 2), or test of subsequent collections (method 3). The three criteria used to distinguish among the methods are reliability, cost, and validity. The pairwise comparison matrix for the three criteria is shown in Table 9.46. The pairwise comparison matrices of the three accounting methods for the three criteria are shown in Table 9.47. Use AHP to determine which auditing procedure should be used. Also check the first pairwise comparison matrix for consistency.

TABLE 9.46 Pairwise Comparisons of Criteria in Accounting Problem

	Reliability	Cost	Validity
Reliability	1	5	7
Cost	1/5	1	2
Validity	1/7	1/2	1

TABLE 9.47 Pairwise Comparisons of Methods in Accounting Problem

Reliability	Method 1	Method 2	Method 3
Method 1	1	1/6	1/2
Method 2	6	1	4
Method 3	2	1/4	1
Cost	Method 1	Method 2	Method 3
Method 1	1	5	3
Method 2	1/5	1	1/2
Method 3	1/3	2	1
Validity	Method 1	Method 2	Method 3
Method 1	1	3	2
Method 2	1/3	1	1/2
Method 3	1/2	2	1

9.5 CONCLUSION

Whenever we face a problem with multiple competing objectives, as is the case in many real-world problems, we are forced to make trade-offs between these objectives. This is usually a very difficult task, and not all management scientists agree on the best way to proceed. When the objectives are very different in nature, no method can disguise the inherent complexity of comparing “apples to oranges.” Although one method, finding Pareto optimal solutions and drawing the resulting trade-off curve, locates solutions that are not dominated by any others, we still face the problem of choosing one of the (many) Pareto optimal solutions to implement. The other two methods we have discussed in this chapter, goal programming and AHP, make trade-offs and ultimately locate an “optimal” solution. These methods are not without their critics, but when they are used carefully, they have the potential to help solve some difficult and important real-world problems.

PROBLEMS

Skill-Building Problems

33. The Pine Valley Board of Education must hire teachers for the coming school year. The types of teachers and the salaries that must be paid are given in Table 9.48. For example, 20 teachers who are qualified to teach history and science have applied for jobs, and each of these teachers must be paid an annual salary of \$21,000. Each teacher who is hired teaches the two subjects he or she is qualified to teach. Pine Valley needs to hire 35 teachers qualified to teach history, 30 teachers qualified to teach science, 40 teachers qualified to teach math, and 32 teachers qualified to teach English. The board has \$1.4 million to spend on teachers’ salaries. A penalty cost of \$1 is incurred for each dollar the board goes over budget. For each teacher by which Pine Valley’s goals are unmet, the following costs are incurred (because of the lower quality of education): science, \$30,000; math, \$28,000; history, \$26,000; and English, \$24,000. Determine how the board can minimize its total cost due to unmet goals.

TABLE 9.48 Data for Pine Valley Problem

Teacher Can Teach	Number Applying	Annual Salary
History and science	20	\$21,000
History and math	15	\$22,000
English and science	12	\$23,000
English and math	14	\$24,000
English and history	13	\$25,000
Science and math	12	\$26,000

34. Stockco fills orders for three products for a local warehouse. Stockco must determine how many of each product should be ordered at the beginning of the current month. This month, 400 units of product 1, 500 units of product 2, and 300 units of product 3 will be demanded. The cost and space taken up by one unit of each product are shown in Table 9.49. If Stockco runs out of stock before the end of the month, the stockout costs shown in Table 9.49 are incurred. Stockco has \$17,000 to spend on ordering products and has 3700 square feet of warehouse space. A \$1 penalty is assessed for each dollar spent over the budget limit, and a \$10 cost is assessed for every square foot of warehouse space needed.

TABLE 9.49 Data for Stockco Problem

	Space (sq ft)	Cost	Stockout Cost
Product 1	6	\$20	\$16
Product 2	5	\$18	\$10
Product 3	4	\$16	\$8

- Determine Stockco’s optimal ordering policy.
- Suppose that Stockco has set the following goals, listed in order of priority:
 - Goal 1: Spend at most \$17,000.
 - Goal 2: Use at most 3700 square feet of warehouse space.
 - Goal 3: Meet demand for product 1.
 - Goal 4: Meet demand for product 2.
 - Goal 5: Meet demand for product 3.
 Develop a preemptive goal programming model for Stockco.

35. BeatTrop Foods is trying to choose one of three companies to merge with. Seven factors are important in this decision:

- Factor 1: Contribution to profitability
- Factor 2: Growth potential
- Factor 3: Labor environment
- Factor 4: R&D ability of company
- Factor 5: Organizational fit
- Factor 6: Relative size
- Factor 7: Industry commonality

The pairwise comparison matrix for these factors is as follows:

$$\begin{bmatrix} 1 & 3 & 7 & 5 & 1 & 7 & 1 \\ 1/3 & 1 & 9 & 1 & 1 & 5 & 1 \\ 1/7 & 1/9 & 1 & 1/7 & 1/5 & 1/2 & 1/4 \\ 1/5 & 1 & 7 & 1 & 1/4 & 7 & 1/3 \\ 1 & 1 & 5 & 4 & 1 & 5 & 3 \\ 1/7 & 1/5 & 2 & 1/7 & 1/5 & 1 & 1/6 \\ 1 & 1 & 4 & 3 & 1/3 & 6 & 1 \end{bmatrix}$$

The three contenders for merger have the following pairwise comparison matrices for each factor:

$$\text{Factor 1: } \begin{bmatrix} 1 & 9 & 3 \\ 1/9 & 1 & 1/5 \\ 1/3 & 5 & 1 \end{bmatrix}$$

$$\text{Factor 2: } \begin{bmatrix} 1 & 7 & 4 \\ 1/7 & 1 & 1/3 \\ 1/4 & 3 & 1 \end{bmatrix}$$

$$\text{Factor 3: } \begin{bmatrix} 1 & 1/5 & 1/3 \\ 5 & 1 & 2 \\ 3 & 1/2 & 1 \end{bmatrix}$$

$$\text{Factor 4: } \begin{bmatrix} 1 & 6 & 3 \\ 1/6 & 1 & 1/2 \\ 1/3 & 2 & 1 \end{bmatrix}$$

$$\text{Factor 5: } \begin{bmatrix} 1 & 1/9 & 1/5 \\ 9 & 1 & 4 \\ 5 & 1/4 & 1 \end{bmatrix}$$

$$\text{Factor 6: } \begin{bmatrix} 1 & 1/7 & 1/4 \\ 7 & 1 & 3 \\ 4 & 1/3 & 1 \end{bmatrix}$$

$$\text{Factor 7: } \begin{bmatrix} 1 & 1/7 & 1/3 \\ 7 & 1 & 3 \\ 3 & 1/3 & 1 \end{bmatrix}$$

Use AHP to determine the company that BeatTrop should merge with.

36. Productco produces three products. Each product requires labor, lumber, and paint. The resource requirements, unit price, and variable cost (exclusive

of raw materials) for each product are given in Table 9.50. At present, 900 labor hours, 1550 gallons of paint, and 1600 board feet of lumber are available. Additional labor can be purchased at \$6 per hour. Additional paint can be purchased at \$2 per gallon. Additional lumber can be purchased at \$3 per board foot. For the following two sets of priorities, use preemptive goal programming to determine an optimal production schedule. For set 1:

- Priority 1: Obtain profit of at least \$10,500.
- Priority 2: Purchase no additional labor.
- Priority 3: Purchase no additional paint.
- Priority 4: Purchase no additional lumber.

For set 2:

- Priority 1: Purchase no additional labor.
- Priority 2: Obtain profit of at least \$10,500.
- Priority 3: Purchase no additional paint.
- Priority 4: Purchase no additional lumber.

TABLE 9.50 Data for Productco Problem

	Labor	Lumber	Paint	Price	Variable
					Cost
Product 1	1.5	2	3	\$26	\$10
Product 2	3.0	3	2	\$28	\$6
Product 3	2.0	4	2	\$31	\$7

Skill-Extending Problems

37. Jobs at Indiana University are rated on three factors:

- Factor 1: Complexity of duties
- Factor 2: Education required
- Factor 3: Mental and/or visual demands

For each job at IU, the requirement for each factor has been rated on a scale of 1–4, with a 4 in factor 1 representing high complexity of duty, a 4 in factor 2 representing high educational requirement, and a 4 in factor 3 representing high mental and/or visual demands. IU wants to determine a formula for grading each job. To do this, it will assign a point value to the score for each factor that a job requires. For example, suppose that level 2 of factor 1 yields a point total of 10, level 3 of factor 2 yields a point total of 20, and level 3 of factor 3 yields a point total of 30. Then a job with these requirements has a point total of $10 + 20 + 30 = 60$. A job's hourly salary equals half its point total. IU has two goals (listed in order of priority) in setting up the points given to each level of each job factor.

- Goal 1: When increasing the level of a factor by 1, the points should increase by at least 10. For example, level 2 of factor 1 should earn at least 10 more points than level 1 of factor 1. Goal 1 is to minimize the sum of deviations from this requirement.

- Goal 2: For the benchmark jobs referred to in Table 9.51, the actual point total for each job should come as close as possible to the point total listed in the table. Goal 2 is to minimize the sum of the absolute deviations of the point totals from the desired scores.

TABLE 9.51 Data for IU Benchmark Jobs

	Factor 1 Level	Factor 2 Level	Factor 3 Level	Desired Score
Job 1	4	4	4	105
Job 2	3	3	2	93
Job 3	2	2	2	75
Job 4	1	1	2	68

Use preemptive goal programming to find appropriate point totals. What salary should a job with skill levels of 3 for each factor be paid?

38. A hospital outpatient clinic performs four types of operations. The profit per operation, as well as the minutes of X-ray time and laboratory time used, are given in Table 9.52. The clinic has 500 private rooms and 500 intensive care rooms. Type 1 and type 2 operations require a patient to stay in an intensive care room for one day, whereas type 3 and type 4 operations require a patient to stay in a private room for one day. Each day the hospital is required to perform at least 100 operations of each type. The hospital has set the following goals (listed in order of priority):
- Goal 1: Earn a daily profit of at least \$100,000.
 - Goal 2: Use at most 50 hours daily of X-ray time.
 - Goal 3: Use at most 40 hours daily of laboratory time.

TABLE 9.52 Data for Hospital Problem

	Type of Operation			
	1	2	3	4
Profit	\$200	\$150	\$100	\$80
X-ray time (minutes)	6	5	4	3
Lab time (minutes)	5	4	3	2

Use preemptive goal programming to determine the types of operations that should be performed.

39. You are trying to determine which city to live in. New York and Chicago are under consideration. Four objectives will determine your decision: housing cost, cultural opportunities, quality of schools and universities, and crime level. The weight for each objective is in Table 9.53. For each objective

(except for quality of schools and universities), New York and Chicago scores are given in Table 9.54. Suppose that the score for each city on the quality of schools and universities depends on two things: a score on public school quality and a score on university quality. The pairwise comparison matrix for public school and university quality is shown in Table 9.55. To see how each city scores on public school quality and university quality, use the pairwise comparison matrices in Table 9.56. You should be able to derive a score for each city on the quality of schools and universities objective. Then use AHP to determine where you should live.

TABLE 9.53 Weights for Objectives in City Problem

Objective	Weight
Housing cost	0.50
Culture	0.10
Education	0.20
Crime	0.20

TABLE 9.54 Scores for Cities in City Problem

	New York	Chicago
Housing cost	0.30	0.70
Culture	0.70	0.30
Crime	0.40	0.60

TABLE 9.55 Pairwise Comparison of Education Objectives in City Problem

	Public School Quality	University Quality
Public school quality	1	4
University quality	1/4	1

TABLE 9.56 Pairwise Comparison of Cities for Education in City Problem

Public School Quality	New York	Chicago
New York	1	4
Chicago	1/4	1

University Quality	New York	Chicago
New York	1	1/3
Chicago	3	1

40. At Lummins Engine Corporation, production employees work ten hours per day, four days per week. Each day of the week, at least the following number of employees must be working: Monday–Friday, seven employees; Saturday and Sunday, three employees. Lummins has set the following goals, listed in order of priority:
- Goal 1: Meet employee requirements with 11 workers.
 - Goal 2: The average number of weekend days off per employee should be at least 1.5 days.
 - Goal 3: The average number of consecutive days off an employee gets during the week should exceed 2.8 days.

Use preemptive goal programming to determine how to schedule Lummins employees.

41. You are the mayor of Gotham City and you must determine a tax policy for the city. Five types of taxes are used to raise money:
- Property taxes. Let p be the property tax rate.
 - A sales tax on all items except food, drugs, and durable goods. Let s be the sales tax rate.
 - A sales tax on durable goods. Let d be the durable goods sales tax rate.
 - A gasoline sales tax. Let g be the gasoline sales tax rate.
 - A sales tax on food and drugs. Let f be the sales tax on food and drugs.

The city consists of three groups of people: low income (LI), middle income (MI), and high income (HI). The amount of revenue (in millions of dollars) raised from each group by setting a particular tax at a 1% level is given in Table 9.57. For example, a 3% tax on durable good sales will raise 360 million dollars from low-income people. Your tax policy must satisfy the following restrictions:

- Restriction 1: The tax burden on MI people cannot exceed \$2.8 billion.
- Restriction 2: The tax burden on HI people cannot exceed \$2.4 billion.
- Restriction 3: The total revenue raised must exceed the current level of \$6.5 billion.
- Restriction 4: s must be between 1% and 3%.

TABLE 9.57 Data for Gotham City Tax Problem

	p	s	d	g	f
LI	900	300	120	30	90
MI	1200	400	100	20	60
HI	1000	250	60	10	40

Given these restrictions, the city council has set the following three goals (listed in order of priority):

- Goal 1: Limit the tax burden on LI people to \$2 billion.
- Goal 2: Keep the property tax rate under 3%.
- Goal 3: If their tax burden becomes too high, 20% of the LI people, 20% of the MI people, and 40% of the HI people may consider moving to the suburbs. Suppose that this will happen if their total tax burden exceeds \$1.5 billion. To discourage this exodus, goal 3 is to keep the total tax burden on these people below \$1.5 billion.

Use preemptive goal programming to determine an optimal tax policy.

42. Based on Sartoris and Spruill (1974). Wivco produces two products, which it sells on both a cash and credit basis. Revenues from credit sales will not have been received but are included in determining profit earned during the current 6-month period. Sales during the next 6 months can be made either from units produced during the next 6 months or from beginning inventory. Relevant information about products 1 and 2 is as follows.

- During the next 6 months, at most 150 units of product 1 can be sold on a cash basis, and at most 100 units of product 1 can be sold on a credit basis. It costs \$35 to produce each unit of product 1, and each sells for \$40. A credit sale of a unit of product 1 yields \$0.50 less profit than a cash sale (because of delays in receiving payment). Two hours of production time are needed to produce each unit of product 1. At the beginning of the 6-month period, 60 units of product 1 are in inventory.
- During the next 6 months, at most 175 units of product 2 can be sold on a cash basis, and at most 250 units of product 2 can be sold on a credit basis. It costs \$45 to produce each unit of product 2, and each sells for \$52.50. A credit sale of a unit of product 2 yields \$1.00 less profit than a cash sale. Four hours of production time are needed to produce each unit of product 2. At the beginning of the 6-month period, 30 units of product 2 are in inventory.
- During the next 6 months, Wivco has 1000 hours for production available. At the end of the next 6 months, Wivco incurs a 10% holding cost on the value of ending inventory (measured relative to production cost). An opportunity cost of 5% is also assessed against any cash on hand at the end of the 6-month period.

- a. Formulate and solve an LP that yields Wivco's maximum profit during the next 6 months. What is Wivco's ending inventory position? Assuming an initial cash balance of \$0, what is Wivco's ending cash balance?
- b. Since an ending inventory and cash position of \$0 is undesirable (for ongoing operations), Wivco is considering other options. At the beginning of the 6-month period, Wivco can

obtain a loan (secured by ending inventory) that incurs an interest cost equal to 5% of the value of the loan. The maximum value of the loan is 75% of the value of the ending inventory. The loan will be repaid one year from now. Wivco has the following goals (listed in order of priority):

- Goal 1: Make the ending cash balance of Wivco come as close as possible to \$75.
- Goal 2: Make profit come as close as possible to the profit level obtained in part a.
- Goal 3: At any time, Wivco's current ratio is defined to be

$$\text{Current ratio} = \frac{\text{Wivco's assets}}{\text{Wivco's liabilities}}$$

Assuming initially that current liabilities equal \$150, 6 months from now Wivco's current ratio will equal

$$\text{Current ratio} = \frac{\text{CB} + \text{AR} + \text{EI}}{150 + \text{Size of loan}}$$

where CB is the ending cash balance, AR is the value of accounts receivable, and EI is the value of the ending inventory. Six months from now, Wivco wants the current ratio to be as close as possible to 2.

Use preemptive goal programming to determine Wivco's production and financial strategy.

Modeling Problems

43. How might you use preemptive goal programming to help Congress balance the budget?
44. A company is considering buying up to five other businesses. Given knowledge of the company's view

of the trade-off between risk and return, how could trade-off curves be used to determine the companies that should be purchased?

45. How would you use AHP to determine the greatest sports record of all time? (Many believe it is Joe Dimaggio's 56-game hitting streak.)
46. You are planning to renovate a hospital. How would you use AHP to help determine what improvements to include in the renovation?
47. You are planning to overhaul a hospital computer system. How would you use AHP to determine the type of computer system to install?
48. You have been commissioned to assign 100 remedial education teachers to the 40 schools in the St. Louis School System. What are some objectives you might consider in assigning the teachers to schools?
49. You have been hired as a consultant to help design a new airport in northern Indiana that will supplant O'Hare as Chicago's major airport. Discuss the objectives you consider important in designing and locating the airport.
50. In the Indiana MBA program we need to divide a class of 60 students into ten six-person teams. In the interest of diversity we have the following goals (listed in descending order of importance):
 - at least one woman per team
 - at least one member of a minority per team
 - at least one student with a financial or accounting background per team
 - at least one engineer per team

Explain how you could use the material in this chapter to develop a model to assign students to teams.

Play Time Toy Company

Play Time Toy faces a highly seasonal pattern of sales. In the past, Play Time has used a *seasonal* production schedule, where the amount produced each month matches the sales for that month. Under this production plan, inventory is maintained at a constant level. The production manager, Thomas Lindop, is proposing a switch to a *level*, or constant, production schedule. This schedule would result in significant savings in production

costs but would have higher storage and handling costs, fluctuating levels of inventories, and would also have implications for financing. Jonathan King, president of Play Time Toy, has been reviewing pro forma income statements, cash budgets, and balance sheets for the coming year under the two production scenarios. Table 9.58 shows the pro forma analysis under seasonal production, and Table 9.59 shows the pro forma analysis under level production.

TABLE 9.58 Seasonal Production

Annual net profit		Play Time Toy Company												
Actual		Projected for 1991												
	Dec 1990	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec	Total
Production (sales value)	850	108	126	145	125	125	125	145	1,458	1,655	1,925	2,057	1,006	9000
Inventory (sales value)	813	813	813	813	813	813	813	813	813	813	813	813	813	
INCOME STATEMENT														
Net sales		108	126	145	125	125	125	145	1,458	1,655	1,925	2,057	1,006	9,000
Cost of goods sold														
Materials & regular wages		70	82	94	81	81	81	94	950	1,079	1,254	1,340	656	5,865
Overtime wages		0	0	0	0	0	0	0	61	91	131	151	0	435
Gross profit		38	44	51	44	44	44	51	447	486	539	565	350	2,700
Operating expenses		188	188	188	188	188	188	188	188	188	188	188	188	2,256
Inventory cost		0	0	0	0	0	0	0	0	0	0	0	0	0
Profit before int & taxes		(150)	(144)	(137)	(144)	(144)	(144)	(137)	259	298	351	377	162	444
Net interest payments		10	2	1	1	2	2	2	3	7	18	19	19	86
Profit before taxes		(160)	(146)	(138)	(146)	(146)	(147)	(140)	256	290	333	359	144	358
Taxes		(55)	(50)	(47)	(50)	(50)	(50)	(48)	87	99	113	122	49	122
Net profit		(106)	(97)	(91)	(96)	(97)	(97)	(92)	169	192	220	237	95	237
BALANCE SHEET														
Cash	175	782	1,365	1,116	934	808	604	450	175	175	175	175	175	
Accts receivable	2,628	958	234	271	270	250	250	270	1,603	3,113	3,580	3,982	3,063	
Inventory	530	530	530	530	530	530	530	530	530	530	530	530	530	
Net P/E	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	
Total Assets	4,403	3,340	3,199	2,987	2,804	2,658	2,454	2,320	3,378	4,888	5,355	5,757	4,838	
Accts payable	255	32	38	44	38	38	38	44	437	497	578	617	302	
Notes payable	680	0	0	0	0	0	0	0	408	1,600	1,653	1,656	966	
Accrued taxes	80	25	(24)	(151)	(232)	(282)	(363)	(411)	(324)	(256)	(143)	(21)	(4)	
Long term debt	450	450	450	450	450	450	425	425	425	425	425	425	400	
Equity	2,938	2,832	2,736	2,644	2,548	2,452	2,355	2,263	2,431	2,623	2,843	3,080	3,175	
Total liab & equity	4,403	3,340	3,199	2,987	2,804	2,658	2,454	2,320	3,378	4,888	5,355	5,757	4,838	

TABLE 9.59
Level Production

Annual net profit		Play Time Toy Company												
	373													
	Actual	Projected for 1991												
	Dec 1990	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec	Total
Production (sales value)	850	750	750	750	750	750	750	750	750	750	750	750	750	9000
Inventory (sales value)	813	1455	2079	2684	3309	3934	4559	5164	4456	3551	2376	1069	813	
INCOME STATEMENT		Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec	Total
Net sales		108	126	145	125	125	125	145	1,458	1,655	1,925	2,057	1,006	9,000
Cost of goods sold														
Materials & regular wages		70	82	94	81	81	81	94	950	1,079	1,254	1,340	656	5,865
Overtime wages		0	0	0	0	0	0	0	0	0	0	0	0	0
Gross profit		38	44	51	44	44	44	51	508	576	671	717	350	3,135
Operating expenses		188	188	188	188	188	188	188	188	188	188	188	188	2,256
Inventory cost		0	2	6	10	13	17	20	16	11	4	0	0	100
Profit before int & taxes		(150)	(147)	(143)	(154)	(158)	(161)	(158)	304	377	478	529	162	779
Net interest payments		10	3	2	5	10	15	21	26	32	37	31	22	214
Profit before taxes		(160)	(149)	(146)	(159)	(168)	(177)	(179)	277	346	441	498	141	565
Taxes		(55)	(51)	(50)	(54)	(57)	(60)	(61)	94	118	150	169	48	192
Net profit		(106)	(99)	(96)	(105)	(111)	(117)	(118)	183	228	291	329	93	373
BALANCE SHEET		Actual	Projected for 1991											
	Dec 1990	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec	
Cash	175	556	724	175	175	175	175	175	175	175	175	175	175	
Accts receivable	2,628	958	234	271	270	250	250	270	1,603	3,113	3,580	3,982	3,063	
Inventory	530	948	1,355	1,749	2,157	2,564	2,971	3,365	2,904	2,314	1,549	697	530	
Net P/E	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	1,070	
Total Assets	4,403	3,533	3,383	3,265	3,672	4,059	4,466	4,880	5,752	6,672	6,374	5,924	4,838	
Accts payable	255	225	225	225	225	225	225	225	225	225	225	225	225	
Notes payable	680	0	0	108	704	1,259	1,900	2,493	3,087	3,693	2,953	2,005	836	
Accrued taxes	80	25	(25)	(155)	(240)	(297)	(389)	(450)	(355)	(269)	(119)	50	66	
Long term debt	450	450	450	450	450	450	425	425	425	425	425	425	400	
Equity	2,938	2,832	2,734	2,637	2,533	2,422	2,305	2,187	2,370	2,599	2,890	3,218	3,311	
Total liab & equity	4,403	3,533	3,383	3,265	3,672	4,059	4,466	4,880	5,752	6,672	6,374	5,924	4,838	

Greg Cole, chief financial officer of Play Time, prepared the two tables. He explained that the pro forma analyses in Tables 9.58 and 9.59 take fully into account the 11% interest payments on the unsecured loan from Bay Trust Company and the 3% interest received from its cash account. An interest charge of 11%/12 on the balance of the loan at the end of a month must be paid the next month. Similarly, an interest payment of 3%/12 on the cash balance at the end of a month is received in the next month.

The inventory available at the end of December 1990 is \$530,000 (measured in terms of cost to produce). Mr. Cole assumed that this inventory

represents a sales value of $\$530,000/0.651667 = \$813,300$.

The inventory and overtime costs in Tables 9.58 and 9.59 are based on the cost information developed by Mr. Lindop. This information is summarized in Table 9.60.

Mr. Cole further explained how the cost information was used in the pro forma analyses. For example, in Table 9.58, the production in August is \$1,458,000. The overtime cost in August is therefore calculated to be \$61,000 ($= 0.15 \times (1,458,000 - 1,049,000)$). Play Time uses LIFO (last-in, first-out) accounting, so overtime costs are always charged in

TABLE 9.60 Play Time Cost Information

- **Gross margin.** The cost of goods sold (excluding overtime costs) is 65.1667% of sales under any production schedule. Materials costs are 30% of sales. All other nonmaterials costs, including regular wages but excluding overtime wages, are 35.1667% of sales.
- **Overtime cost.** Running at capacity but without using any overtime, the plant can produce \$1,049,000 of monthly sales. Units produced in excess of this capacity in a month incur an additional overtime cost of 15% of sales. (The monthly production capacity of the plant running on full overtime is \$2,400,000 of sales. Since November has the maximum level of projected sales at \$2,057,000, the capacity on full overtime should never pose a problem.)
- **Inventory cost.** The plant has a limited capacity to store finished goods. It can store \$1,663,000 worth of sales at the plant. Additional units must be moved and stored in rented warehouse space. The cost of storage, handling, and insurance of finished goods over this capacity is 7% of the sales value of the goods per year, or 7%/12 per month.

the month that they occur.² The annual overtime cost for the seasonal production plan is \$435,000. In Table 9.59, under level production, finished goods worth \$5,164,000 are in inventory at the end of July. The inventory cost for the month is \$20,000 ($= 0.07/12 \times (5,164,000 - 1,663,000)$). The annual inventory cost for the level production plan is \$100,000.

Mr. Lindop felt that a minimum of \$813,300 of inventory (measured in terms of sales value, or \$530,000 measured in terms of cost to produce) must be kept on hand at the end of each month. This

²This assumes that overtime production is used only to satisfy current demand and not to build up inventory.

inventory level represents a reasonable safety stock, required since orders do not occur uniformly during a month.

Mr. King was impressed at the possible increase in profit from \$237,000 under the seasonal production plan to \$373,000 under level production. While studying the pro forma projections, Mr. King realized that some combination of the two production plans might be even better. He asked Mr. Lindop to try to find a production plan with a higher profit than the seasonal and level plans.

Mr. Lindop proceeded to develop a spreadsheet-based linear programming model to maximize annual net profit.

Questions

Note: Mr. Lindop's model is contained in the file PLAYTIME.XLS. The spreadsheet is ready to be optimized, but it has not been optimized yet.

1. Run the optimization model in the file PLAYTIME.XLS. What is the optimal production plan? What is the optimal annual net profit? How does this optimal production plan compare to the seasonal and level production plans?
2. Suppose that Play Time's bankers will not extend any credit over \$1.9 million—in other words, the loan balance in any month cannot exceed \$1.9 million. Modify the spreadsheet model to take into account this restriction. What is the optimal production plan in this case? What is the optimal annual net profit?
3. Annual profit is a measure of reward for Play Time Toy. The maximum loan balance is a measure of risk for the bank. Construct a trade-off curve between optimal annual profit and the maximum loan balance.